HOMOGENIZATION AND PHASE SEPARATION WITH FIXED WELLS - THE SUPERCRITICAL CASE

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Abstract. A variational model for the interaction between homogenization and phase separation is considered in the regime where the former happens at a smaller scale than the latter. The first order $\Gamma-$limit is proven to exhibit a separation of scales which has only been previously conjectured.

1. Introduction

Composite materials are becoming more important to modern technology as the mixing of two different material properties at fine scales can give rise to unexpected emergent behavior [29][33]. Thus, understanding the process of phase separation on such a material is crucial to leveraging these processes for technological applications.

For a homogeneous material, the distribution of stable phases is commonly modeled by using the Cahn-Hilliard free energy (also known as the Modica-Mortola functional, in the mathematical community). The energy reads as

$$E_\varepsilon(u) := \int_\Omega W\left(u(x)\right) + \varepsilon^2 |\nabla u(x)|^2 \, dx,$$

where $u \in W^{1,2}(\Omega; \mathbb{R}^M)$ represents the distribution of phases, $\varepsilon > 0$ is a small parameter, and the potential $W : \mathbb{R}^M \to [0, \infty)$ vanishes at the stable critical phases. It was first proved by Modica and Mortola [27][28] in the scalar case that this energy minimizes perimeter in the limit. This sharp interface limit was conjectured by Gurtin [21] to hold in more generality and was later proven in [24, 30, 19]. Since then, many variants have been studied such as with multiple phases [7], fully coupled singular perturbations [8, 18], and even the case of when the wells of W are allowed to depend on position [15, 13].

Our interest is in a heterogeneous material where the heterogeneities are modeled with an oscillating periodic potential $W : \mathbb{R}^N \times \mathbb{R}^M \to [0, +\infty)$ that is Q-periodic in the first variable, where $Q \subset \mathbb{R}^N$ is the unit cube. Furthermore, in this paper we consider the wells of W to be fixed constants $a, b \in \mathbb{R}^M$ and the case of when wells of W are dependent on the spatial variable will be considered in future work.

We consider the functional

$$\mathcal{F}_{\varepsilon,\delta}(u) := \int_\Omega W\left(\frac{x}{\delta}, u\right) + \varepsilon^2 |\nabla u|^2 \, dx.$$

In order to analyze the behavior of minimizers, we use the technique of $\Gamma-$expansion [10, 4]. Using standard homogenization techniques, $F_{\varepsilon,\delta} \xrightarrow{\Gamma} F_0$ with

$$F_0(u) = \int_\Omega W_r(u) \, dx$$

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where $u \in L^2(\Omega\ \mathbb{R}^M)$ and W_r is a homogenized potential whose form depends on the rate of convergence $r := \lim \frac{\delta_n}{\varepsilon_n}$. Since we are in the regime of fixed wells a, b, it is always possible to find many minimizers of F_0 which achieve zero energy even with a mass constraint. Thus, in order to better understand the minimizers, we need to consider the next order in the $\Gamma-$expansion. Similar to the heuristics for the homogeneous Modica-Mortola functional, ìt is possible to determine that the energy of having a transition layer between the phases will be of order ε. This leads us to consider the rescaled functional

$$\widetilde{\mathcal{E}}_{\varepsilon,\delta}(u) \coloneqq \frac{1}{\varepsilon}\mathcal{F}_{\varepsilon,\delta}(u) = \int_\Omega \frac{1}{\varepsilon} W\left(\frac{x}{\delta}, u\right) + \varepsilon|\nabla u|^2\, dx \tag{1.1}$$

However, this energy has not been studied much in the literature due to technical mathematical difficulties it poses. The behavior of minimizers of such a model depends greatly on the rate at which ε, δ comparatively decay to 0, i.e. $\varepsilon \ll \delta, \varepsilon \sim \delta$, or $\delta \ll \varepsilon$. The first two authors in collaboration with Hagerty and Popovici, in [14] rigorously characterized the first order Gamma-limit when $\varepsilon \sim \delta$. This has been recently extended to the fully coupled scalar case with stochastic homogenization [26]. For $\varepsilon \ll \delta$, the characterization of the $\Gamma-$limit is still open, but in [13], the authors have identified an intermediate scaling of $\frac{\varepsilon}{\delta}$ and identified a $\Gamma-$limit with respect to strong two-scale convergence in an analysis which also extends to the case of spatially dependent phases.

In this paper we study the case $\delta \ll \varepsilon$, and we prove a separation of scales that has only been conjectured, namely that the first order Gamma-limit is the Gamma-limit of the functional

$$\int_\Omega \frac{1}{\varepsilon} W_{\text{hom}}(u(x)) + \varepsilon|\nabla u(x)|^2\, dx,$$

where W_{hom} is the homogenized potential of W, defined as

$$W_{\text{hom}}(p) \coloneqq \int_Q W(y,p)dy.$$

Heuristically, this is expected because, the regime $\delta \ll \varepsilon$ suggests that we first homogenize (namely, we first send $\delta \to 0$), and then we do phase separation (namely, we send $\varepsilon \to 0$). Indeed, [22] was able to use the technique of direct replacement of the potential by W_{hom} (first used in [11] in a similar setting) to show the $\Gamma-$limit in the regime of $\delta \ll \varepsilon^{\frac{3}{2}}$. Here, we are able to prove that the same heuristics can be made rigorous even when $\delta \ll \varepsilon$, by using a different strategy that relies on the use of the two-scale unfolding of the functional.

Our strategy which will be outlined in Section 1.2 also enables us to weaken the regularity properties of W to allow it to be only Carathéodory and to remove restrictions such as quadratic behavior near the wells which are sometimes used in literature [13]. We also highlight that our strategy allows us to prove strong compactness directly from the coercivity and polynomial growth bounds rather than assuming the existence of a continuous double well potential independent of x, i.e. a $W_H(s) \leq W(x,z)$ for every $z \in \mathbb{R}^M$ such as used in [22].

Finally, we note that sometimes in the literature, the heterogeneity is entered into the energy through the singular perturbation [2][3]. This creates a similar separation of scales effect, but it requires different techniques and results in different effective limits. This work has since also been extended to the case of the Ambrosio-Totorelli energy [6][5].

1.1. **Main results.** Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, and $N, M \geq 1$. Denote by $Q \coloneqq (-1/2, 1/2)^N$ the unit cube in $\mathbb{R}^N$ centered at the origin. Let $W : \mathbb{R}^N \times \mathbb{R}^M \to [0, +\infty)$ be a measurable function which satisfies the following:

(W1) W is a Carathéodory function which is Q−periodic in the spatial variable, *i.e.*,
- $z \mapsto W(x, z)$ is continuous for $\mathcal{L}^N$ a.e $x \in Q$
- $x \mapsto W(x, z)$ is measurable and $Q-$ periodic for all $z \in \mathbb{R}^M$

(W2) There are $a, b \in \mathbb{R}^M$ such that

$$W(x, z) = 0 \iff z \in \{a, b\}$$

(W3) There exists $R_1 > 0$ such that for $\mathcal{L}^N$-a.e. $x \in Q$,

$$W(x, z) \geq \frac{1}{R_1}|z|^2,$$

if $|z| \geq R_1$, and

$$W(x, z) \leq R_1(1 + |z|^2),$$

for every $z \in \mathbb{R}^M$.

Remark 1.1. We note that we use two-wells and quadratic growth for convenience. Our results will still hold with multiple wells and $p \geq 2$.

We now introduce the functionals that we will study.

Definition 1.2. Let $\{\varepsilon_n\}_n, \{\delta_n\}_n$ be infinitesimal sequences such that

$$\lim_{n\to\infty} \frac{\delta_n}{\varepsilon_n} = 0.$$

For $n \in \mathbb{N}$, define the functional $\mathcal{E}_n : L^2(\Omega; \mathbb{R}^M) \to [0, +\infty]$ as

$$\mathcal{E}_n(u) := \begin{cases} \displaystyle\int_\Omega \left[\frac{1}{\varepsilon_n} W\left(\frac{x}{\delta_n}, u(x)\right) + \varepsilon_n |\nabla u(x)|^2 \right] dx & \text{if } u \in W^{1,2}(\Omega; \mathbb{R}^M), \\ +\infty & \text{else.} \end{cases}$$

Finally, we introduce the limiting functional.

Definition 1.3. For $z \in \mathbb{R}^M$, let

$$W_{\mathrm{hom}}(z) := \int_Q W(x, z)\, dx.$$

Set

$$\sigma_{\mathrm{hom}} := \inf\left\{ \int_{-1}^1 2\sqrt{W_{\mathrm{hom}}(\gamma(t))}|\gamma'(t)|dt : \gamma \in \mathrm{Lip}_{\mathcal{Z}}([-1, 1]; \mathbb{R}^M),\ \gamma(-1) = a,\ \gamma(1) = b \right\},$$

where $\mathrm{Lip}_{\mathcal{Z}}([-1, 1]; \mathbb{R}^M)$ is the space of continuous curves $\gamma : [-1, 1] \to \mathbb{R}^M$ such that $\gamma \in \mathrm{Lip}(T, \mathbb{R}^M)$, for every compact set $T \subset [-1, 1]$ disjoint from $\{t \in [-1, 1] : \gamma(t) \in \{a, b\}\}$.

Define the functional $\mathcal{E}_\infty : L^2(\Omega; \mathbb{R}^M) \to [0, +\infty]$ as

$$\mathcal{E}_\infty(u) := \begin{cases} \sigma_{\mathrm{hom}} \mathrm{Per}(\{u = a\}) & \text{if } u \in BV(\Omega; \{a, b\}), \\ +\infty & \text{else.} \end{cases}$$

We are now in position to state the two main results of this paper, namely pre-compactness of sequences with uniformly bounded energy, and the Γ-convergence of $\{\mathcal{E}_n\}_n$.

Theorem 1.4. *Let $\{\varepsilon_n\}_n, \{\delta_n\}_n$ be infinitesimal sequences such that*

$$\lim_{n\to\infty} \frac{\delta_n}{\varepsilon_n} = 0.$$

Let $\{u_n\}_n \subset L^2(\Omega;\mathbb{R}^M)$ be such that

$$\sup_{n\in\mathbb{N}} \mathcal{E}_n(u_n) < \infty.$$

Then, there exist $u \in BV(\Omega;\{a,b\})$ and a subsequence $\{u_{n_k}\}_k$ such that $u_{n_k} \to u$ strongly in $L^2(\Omega;\mathbb{R}^M)$.

Theorem 1.5. *Let $\{\varepsilon_n\}_n, \{\delta_n\}_n$ be infinitesimal sequences such that*

$$\lim_{n\to\infty} \frac{\delta_n}{\varepsilon_n} = 0.$$

Then, $\mathcal{E}_n \xrightarrow{\Gamma} \mathcal{E}_\infty$ with respect to strong $L^2(\Omega;\mathbb{R}^M)$ convergence.

The strategy of the proofs are stable enough to allow for a mass constraint to be incorporated in the functional.

Corollary 1.6. *Let $m \in (0,|\Omega|)$. Define*

$$\widetilde{\mathcal{E}}_n(u) := \begin{cases} \mathcal{E}_n(u) & \text{if } u \in W^{1,2}(\Omega;\mathbb{R}^M),\ \displaystyle\int_\Omega u\,dx = ma + (1-m)b, \\ +\infty & \text{else.} \end{cases}$$

Let $\{\varepsilon_n\}_n, \{\delta_n\}_n$ be infinitesimal sequences such that

$$\lim_{n\to\infty} \frac{\delta_n}{\varepsilon_n} = 0.$$

Then, $\widetilde{\mathcal{E}}_n \xrightarrow{\Gamma} \widetilde{\mathcal{E}}_\infty$ with respect to strong $L^2(\Omega;\mathbb{R}^M)$ convergence, where

$$\widetilde{\mathcal{E}}_\infty(u) := \begin{cases} \mathcal{E}_\infty(u), & \text{if } u \in BV(\Omega;\{a,b\}),\ |\{u=a\}| = m, \\ +\infty & \text{else.} \end{cases}$$

Moreover, pre-compactness for sequences with uniformly bounded $\widetilde{\mathcal{E}}_n$ energy holds.

Remark 1.7. The Gamma-convergence results stated in Theorem 1.5 and Corollary 1.6 allow to get the standard convergence of minima and minimizers (see Theorem 2.7), as well as approximation of isolated local minimizers (see Theorem 2.8).

1.2. **Outline of the Strategy.** The recovery sequence is the same recovery sequence as for that for the Modica-Mortola energy with potential W_{hom}. The recovery sequence and the modifications required to satisfy the usual mass constraint are reported in Sections 6 and 7 which require some care due to the minimal assumptions on our W. The core of the work is in proving the Liminf inequality in Section 4 with some useful preliminary and auxiliary results contained in Sections 2 and 3. Here we outline the main ideas of the proof of the Liminf inequality.

Take a sequence $\{u_n\} \subset W^{1,2}(\Omega;\mathbb{R}^M) \cap L^\infty(\Omega\,\mathbb{R}^M)$ with bounded energy which achieves the Liminf. We can partially unfold the energy with the unfolding operator (see Definition 2.1) just on the potential and use the non-negativity to throw away the boundary terms at the cost of the correct inequality.

$$\mathcal{E}_n[u_n] \geq \int\limits_\Omega \int\limits_Q \frac{W(y, \mathcal{U}_{\delta_n} u_n)}{\varepsilon_n}\, dy + \varepsilon_n |\nabla u_n|^2 \, dx \tag{1.2}$$

Now apply the Modica-Mortola trick to get:

$$\mathcal{E}_n[u_n] \geq \int_\Omega 2 \left[\int_Q W(y, \mathcal{U}_{\delta_n} u_n(x,y))\, dy \right]^{\frac{1}{2}} |\nabla u_n| \, dx =: F_n[u_n]$$

To finish we would need to replace the integral under the square root sign by

$$W_{\text{hom}}(u_n) = \int_Q W(y, u_n)\, dy.$$

First we notice that the following term is negligible in the limit (see (3.5)).

$$\int_\Omega \int_Q |\mathcal{U}_{\delta_n} u_n - u_n| \, dy |\nabla u_n| \, dx \to 0.$$

Rewriting F_n as,

$$\int_\Omega 2 \left[\int_Q W(y, u_n(x) + \mathcal{U}_{\delta_n} u_n(x,y) - u_n(x))\, dy \right]^{\frac{1}{2}} |\nabla u_n| \, dx,$$

a suitable idea is to claim that the potential essentially acts like $W_{\text{hom}}(u_n)$, but with the exception of some small sets.

So we embark on the following program. We fix an $\eta > 0$. Then using a slicing argument, we find a sequence $\{v_n^\eta\}_n \subset L^\infty(\Omega; W_0^{1,2}(Q))$ with $\|v_n\|_\infty \leq \eta$ and lower energy:

$$\liminf_{n\to\infty} F_n[u_n] \geq \liminf_{n\to\infty} \int_\Omega 2 \left[\int_Q W(y, u_n + v_n^\eta)\, dy \right]^{\frac{1}{2}} |\nabla u_n| \, dx$$

Then we appropriately define a double-well function $W^\eta(p)$ (see Section 3.2) such that v_n^η is admissible in the infimum. After some technicalities regarding truncations and taking out appropriately defined small sets, we achieve

$$\liminf_{n\to\infty} F_n[u_n] \geq \liminf_{n\to\infty} \int_\Omega 2\sqrt{W^\eta(u_n)} |\nabla u_n| \, dx.$$

Since W^η is still a continuous double-well function with the same wells (see Theorem 3.4), we can apply the classical compactness and Liminf arguments [19] to achieve

$$\liminf_{n\to\infty} F_n[u_n] \geq \sigma_\eta \mathrm{Per}(\{u_0 = a\}).$$

To conclude, we show that $\sigma_\eta \nearrow \sigma_{\text{hom}}$ where σ_{hom} is as defined in Definition 1.3.

Here we use the ideas from [34] where Zuniga and Sternberg showed under very minimal conditions the existence of a minimizer of the geodesic problems that form σ_η. We recall these properties in Section 2.3 and use them to prove some critical results in Section 3.3.

Below we will detail the results, and we note that we will often take subsequences without relabeling and C will be a generic constant that may change between inequalities and subscripts to C will describe the limiting parameters that it depends upon.

2. Preliminaries

2.1. **The unfolding operator.** We recall the unfolding operator which was first used to more easily define two scale convergence [12][31][32]. While we do not need two scale convergence in this paper, it is still a nice tool to encode the usual change of variable used in homogenization problems.

Definition 2.1. For $\delta > 0$, let

$$\hat{\Omega}_\delta := \bigcup_{z_i \in I_\delta} \left(\overline{z_i + \delta Q}\right) \cap \Omega, \qquad \Lambda_\delta := \Omega \setminus \hat{\Omega}_\delta,$$

where I_δ is the set of points $z_i \in \delta\mathbb{Z}^N$ such that $\overline{z_i + \delta Q} \subset \Omega$. The unfolding operator $\mathcal{U}_\delta : L^2(\Omega;\mathbb{R}^M) \to L^2(\Omega; L^2(Q;\mathbb{R}^M))$ is defined as

$$\mathcal{U}_\delta(u)(x,y) := \begin{cases} u\Big(\delta \left\lfloor \frac{x}{\delta} \right\rfloor + \delta y\Big) & \text{for } x \in \hat{\Omega}_\delta,\ y \in Q, \\ a & \text{if } x \in \Lambda_\delta, y \in Q, \end{cases} \tag{2.1}$$

where, given an enumeration $\{z_i\}_{i\in\mathbb{N}}$ of $\mathbb{Z}^N$,

$$\lfloor x \rfloor := z_i \qquad i := \min\big\{ j \in \mathbb{N} \,:\, z_j \in \operatorname{argmin}\{|z - x| : z \in \mathbb{Z}^N\} \big\} \tag{2.2}$$

is the integer part of $x \in \mathbb{R}^N$, and a is the well in (W2).

Remark 2.2. This definition of the unfolding operator is nonstandard as we make the unfolding operator nonzero in the small boundary set $\Lambda_\delta \times Q$. This has been used previously in [13].

2.2. **Truncation of functions.** Finally, we define the truncated operator and we state its basic properties.

Definition 2.3. For $M > 0$, we define the truncation operator $\mathcal{T}_M : L^2(\Omega;\mathbb{R}^M) \to L^\infty(\Omega;\mathbb{R}^M)$ as

$$\mathcal{T}_M(f)(x) := \begin{cases} f(x) & |f(x)| \le M, \\ M\dfrac{f(x)}{|f(x)|} & |f(x)| > M. \end{cases}$$

Lemma 2.4. *Let $f \in W^{1,2}(\Omega;\mathbb{R}^M)$. Then, $\mathcal{T}_M(f) \in L^\infty(\Omega;\mathbb{R}^M) \cap W^{1,2}(\Omega;\mathbb{R}^M)$ and $|\nabla \mathcal{T}_M(f)| \le |\nabla f|$.*

2.3. **Geodesics of degenerate metrics.** Here we describe the results from [34], which will be used extensively to prove convergence of the degenerate geodesic problems without using many assumptions.

Let $F : \mathbb{R}^M \to [0,\infty)$ be a continuous function satisfying

(F1) The zero set of F, denoted by $\mathcal{Z}$, consists of a finite number of distinct points;

(F2) $\liminf_{|z|\to\infty} F(z) > 0$.

First, we define $\mathrm{Lip}_{\mathcal{Z}}([0,1];\mathbb{R}^M)$ to be the space of continuous curves which are locally Lipschitz continuous with respect to the Euclidean metric on any portion of the curve which does not touch the zero set of F.

Consider the energy

$$E(\gamma) := \int_{-1}^{1} F(\gamma(t))|\gamma'(t)|dt.$$

Due to the parameterization invariance of the energy and the fact that it is conformal to the Euclidean metric up to a degenerate factor, we can define a metric on $\mathbb{R}^M$ by

$$d(p,q) := \inf\left\{\int_0^1 F(\gamma(t))|\gamma'(t)|dt : \gamma \in \mathrm{Lip}_{\mathcal{Z}}([0,1];\mathbb{R}^M),\ \gamma(0) = p,\ \gamma(1) = q\right\},$$

and $(\mathbb{R}^M, d)$ is a length space.

We can also define the length functional L for any curve γ

$$L(\gamma) := \sup_{\{t_k\}_k \subset \mathcal{P}} \sum_k d(\gamma(t_k), \gamma(t_{k+1})),$$

where $\mathcal{P}$ is the set of finite partitions of $[-1,1]$. Using this length space viewpoint, it is proven in [34] that

Proposition 2.5.

(1) [34, Lemma 2.4] *Let B(x,r) denote the open ball centered at x and with radius r in the Euclidean metric. For every $\varepsilon > 0$ such that $\mathcal{Z} \subset B(0, \frac{1}{\varepsilon})$, there is an $r_\varepsilon > 0$ such that if $p, q \in B(0, \frac{1}{\varepsilon}) \cap \left(\bigcup_{z\in\mathcal{Z}} B(z, 2\varepsilon)^c\right)$ then there is a d-minimizing curve, $\gamma^* \in \mathrm{Lip}_{\mathcal{Z}}([0,1];\mathbb{R}^M)$ such that*

$$\gamma^*([0,1]) \cap \left(\bigcup_{z\in\mathcal{Z}} B(z,\varepsilon)\right) = \emptyset;$$

(2) [34, Theorem 2.5] *For any $\gamma \in \mathrm{Lip}_{\mathcal{Z}}([0,1];\mathbb{R}^M)$, we have*

$$L(\gamma) = E(\gamma);$$

(3) [34, Theorem 2.6] *For every $p, q \in \mathbb{R}^M$, there is a minimizing $\gamma^* \in \mathrm{Lip}_{\mathcal{Z}}([0,1];\mathbb{R}^M)$ which satisfies*

$$d(p,q) = E(\gamma^*) = L(\gamma^*);$$

(4) [34, Proposition 2.7] *Furthermore, given any partition $\{t_k\}$ of $[0,1]$, a minimizer $\gamma^* \in \mathrm{Lip}_{\mathcal{Z}}([0,1];\mathbb{R}^M)$ satisfies*

$$L(\gamma^*) = \sum_k d(\gamma^*(t_k), \gamma^*(t_{k+1})).$$

2.4. **Γ-convergence.** In this section, we recall the definition and the basic properties of Gamma-limits. Since in this paper we work in the setting of the metric space $L^2(\Omega\mathbb{R}^M)$, we will present the equivalent definition with sequences. We refer to [16] (see also [9]) for a complete study of Γ-convergence on topological spaces.

Definition 2.6. Let (X, d) be a metric space, and let $\{F_n\}_n$ be a sequence of functionals $F_n : X \to [-\infty, +\infty]$. We say that $\{F_n\}_n$ Γ-converges to $F : X \to [-\infty, +\infty]$ with respect to the metric d, if the followings hold:

(i) For every $x \in X$ and every $\{x_n\}_n \subset X$ with $x_n \to x$, we have

$$F(x) \le \liminf_{n\to\infty} F_n(x_n)\,,$$

(ii) For every $x \in X$, there exists $\{x_n\}_n \subset X$ (the so called *recovery sequence*) with $x_n \to x$, such that

$$\limsup_{n\to\infty} F_n(x_n) \leq F(x)\,.$$

The definition of Γ-convergence was designed to characterize in a variational way the limiting behaviour of sequences of global minimizers, as well as of the minima (see, for example, [16, Corollary 7.20]).

Theorem 2.7. *Let (X, d) be a metric space. Consider, for each $n \in \mathbb{N}$, a functional $F_n : X \to \mathbb{R} \cup \{\infty\}$, and assume that the sequence $\{F_n\}_n$ Γ-converges to some $F : X \to \mathbb{R} \cup \{\infty\}$. For each $n \in \mathbb{N}$, let $x_n \in X$ be a minimizer of F_n on X. Then, every cluster point $x \in X$ of $\{x_n\}_n$ is a minimizer of F, and*

$$F(x) = \limsup_{n\to\infty} F_n(x_n).$$

If the point $x \in X$ is a limit of the sequence $\{x_n\}_n$, then the limsup is actually a limit.

In general, it is not possible to obtain every minimum point of the limiting functional as limit of minimizers of the F_n's (although, it is possible to obtain them as limits of ε-minimizers, see [16, Theorem 7.12]). Nevertheless, it is possible to obtain an approximation result for *isolated* local minimizers of the limiting functional. This was proved by Kohn and Sternberg in [24].

Theorem 2.8. *Let (X, d) be a metric space. Consider, for each $n \in \mathbb{N}$, a functional $F_n : X \to \mathbb{R} \cup \{\infty\}$, and assume that the sequence $\{F_n\}_n$ Γ-converges to some $F : X \to \mathbb{R} \cup \{\infty\}$. Let $x \in X$ be an isolated local minimizer for F. Then, there exists a sequence $\{x_n\}_n \subset X$, where, for each $n \in \mathbb{N}$, x_n is a local minimizer of F_n, such that $x_n \to x$.*

2.5. **Sets of finite perimeter.** We recall the definition and some basic facts about sets of finite perimeter that we need in the paper. For more details on the subject, we refer the reader to standard references, such as [1, 17, 20, 25].

Definition 2.9. Let $E \subset \mathbb{R}^N$ with $|E| < \infty$, and let $A \subset \mathbb{R}^N$ be an open set. We say that E has *finite perimeter* in A if

$$P(E; A) := \sup\left\{ \int_E \mathrm{div}\varphi\, dx \,:\, \varphi \in C^1_c(A; \mathbb{R}^N)\,,\, \|\varphi\|_{L^\infty} \leq 1 \right\} < \infty.$$

Definition 2.10. Let $a, b \in \mathbb{R}^M$. We define the space $BV(\Omega; \{a, b\})$ as the space of functions $u \in L^1(\Omega; \mathbb{R}^M)$ with $u(x) \in \{a, b\}$ for a.e. $x \in \Omega$, and such that the set $\{x \in \Omega : u(x) = a\}$ has finite perimeter in Ω.

Definition 2.11. Let $E \subset \mathbb{R}^N$ be a set of finite perimeter in the open set $A \subset \mathbb{R}^N$. We define $\partial^* E$, the *reduced boundary* of E, as the set of points $x \in \mathbb{R}^N$ for which the limit

$$\nu_E(x) := -\lim_{r\to 0} \frac{D\chi_E(B(x,r))}{|D\chi_E|(B(x,r))}$$

exists and is such that $|\nu_E(x)| = 1$. The vector $\nu_E(x)$ is called the *measure theoretic exterior normal* to E at x.

We recall part of the De Giorgi's structure theorem for sets of finite perimeter.

Theorem 2.12. *Let $E \subset \mathbb{R}^N$ be a set of finite perimeter in the open set $A \subset \mathbb{R}^N$. Then,*

$$P(E, B) = \mathcal{H}^{N-1}(\partial^* E \cap B),$$

for all Borel sets $B \subset A$.

3. Technical results

In this section we collect the main technical results that will be used in the proofs of the main theorems. There are two reasons why we isolate these results from the rest of the argument: first of all it makes the arguments more readable and easy to follow, since we can focus on the main ideas and refer to this section for some technical details. Moreover, the results we present here have an interest on their own, and therefore the interested reader can read them more easily.

3.1. **Estimates for sequences with uniformly bounded energy.** We start by investigating bounds that sequences with uniformly bounded energy satisfy. These will be used to compare the energy of a sequence $\{u_n\}_n$ with the energy of the sequence $\{T_{\delta_n u_n}\}_n$. The first result follows directly from the definition of the unfolding operator.

Theorem 3.1. *Let $\{u_n\}_n \subset W^{1,2}(\Omega;\mathbb{R}^M)$ be such that*

$$\sup_{n\in\mathbb{N}} \mathcal{E}_n(u_n) \le C.$$

Then,

$$\|\nabla u_n\|^2_{L^2(\Omega;\mathbb{R}^{N\times M})} \le C\frac{1}{\varepsilon_n}. \tag{3.1}$$

Rewriting in terms of the unfolding operator, we have

$$\int_\Omega \int_Q \frac{W\left(y, \mathcal{U}_{\delta_n} u_n\right)}{\varepsilon_n} + \frac{\varepsilon_n}{\delta_n^2}|\nabla_y \mathcal{U}_{\delta_n} u_n|^2 \, dy \, dx \le \mathcal{E}_n(u_n). \tag{3.2}$$

which directly implies,

$$\|\nabla_y \mathcal{U}_{\delta_n} u_n\|^2_{L^2(\Omega;L^2(Q;\mathbb{R}^{N\times M}))} \le C\frac{\delta_n^2}{\varepsilon_n}. \tag{3.3}$$

We also obtain a second result by slightly modifying the results in [11][22] rewritten in terms of the unfolding operator.

Theorem 3.2. *Let $\{u_n\}_n \subset W^{1,2}(\Omega;\mathbb{R}^M)$ be such that*

$$\sup_{n\in\mathbb{N}} \mathcal{E}_n(u_n) \le C.$$

Then,

$$\|\mathcal{U}_{\delta_n} u_n - u_n\|^2_{L^2(\Omega;L^2(Q;\mathbb{R}^M))} \le C\left(\|\nabla_y \mathcal{U}_{\delta_n} u_n\|^2_{L^2(\Omega;L^2(Q;\mathbb{R}^{N\times M}))} + \int_{\Omega\setminus\hat{\Omega}_{\delta_n}} |u_n - a|^2 dx\right). \tag{3.4}$$

Moreover, if $\sup_n \|u_n\|_\infty < +\infty$, we have

$$\|\mathcal{U}_{\delta_n} u_n - u_n\|_{L^2(\Omega;L^2(Q;\mathbb{R}^M))}\|\nabla u_n\|_{L^2(\Omega;\mathbb{R}^{N\times M})} \le C\left(\frac{\delta_n^2}{\varepsilon_n^2} + \frac{\delta_n}{\varepsilon_n}\right)^{\frac{1}{2}}. \tag{3.5}$$

Proof. **Step 1.** We first prove (3.4). For $x \in \Omega$, let

$$(\mathcal{U}_{\delta_n} u_n)_Q(x) := \int_Q \mathcal{U}_{\delta_n} u_n(x,y)\, dy.$$

By using the triangle inequality, together with the inequality $(p+q)^2 \leq 2(p^2+q^2)$, we get

$$\|\mathcal{U}_{\delta_n} u_n - u_n\|^2_{L^2} \leq 2\|\mathcal{U}_{\delta_n} u_n - (\mathcal{U}_{\delta_n} u_n)_Q\|^2_{L^2} + 2\|u_n - (\mathcal{U}_{\delta_n} u_n)_Q\|^2_{L^2}. \tag{3.6}$$

where the norm is the $L^2(\Omega; L^2(Q;\mathbb{R}^M))$ norm. First, we estimate the latter term on the right-hand side of (3.6). Writing

$$\int_\Omega |u_n - (\mathcal{U}_{\delta_n} u_n)_Q|^2 dx = \int_{\hat{\Omega}_{\delta_n}} |u_n - (\mathcal{U}_{\delta_n} u_n)_Q|^2 dx + \int_{\Omega \setminus \hat{\Omega}_{\delta_n}} |u_n - a|^2 dx,$$

and using the fact that $\mathcal{U}_{\delta_n}[(\mathcal{U}_{\delta_n} u_n)_Q] = (\mathcal{U}_{\delta_n} u_n)_Q$, we get

$$\int_\Omega |u_n - (\mathcal{U}_{\delta_n} u_n)_Q|^2 dx \leq \|\mathcal{U}_{\delta_n} u_n - (\mathcal{U}_{\delta_n} u_n)_Q\|^2_{L^2(\Omega;L^2(Q;\mathbb{R}^M))} + \int_{\Omega \setminus \hat{\Omega}_{\delta_n}} |u_n - a|^2 dx. \tag{3.7}$$

Now, we estimate the first term on the right-hand side of (3.6). By the Poincairé-Wirtinger inequality in the y-variable, for each $x \in \Omega$, we get

$$\int_Q |\mathcal{U}_{\delta_n} u_n - (\mathcal{U}_{\delta_n} u_n)_Q|^2 \, dy \leq C \int_Q |\nabla_y \mathcal{U}_{\delta_n} u_n|^2 \, dy.$$

Integrating over Ω we get the bound

$$\|\mathcal{U}_{\delta_n} u_n - (\mathcal{U}_{\delta_n} u_n)_Q\|^2_{L^2(\Omega;L^2(Q;\mathbb{R}^M))} \leq C\|\nabla_y \mathcal{U}_{\delta_n} u_n\|^2_{L^2(\Omega;L^2(Q;\mathbb{R}^{N\times M}))}. \tag{3.8}$$

Thus, from (3.6), (3.7), and (3.8), we deduce (3.4).

Step 2. We now prove (3.5). By using the fact that $\sup_n \|u_n\|_\infty < +\infty$, we get

$$\int_{\Omega \setminus \hat{\Omega}_{\delta_n}} |u_n - a|^2 dx \leq C|\Omega \setminus \hat{\Omega}_{\delta_n}| \leq C\delta_n.$$

Using (3.4) together with (3.1), (3.3), we conclude that

$$\|\mathcal{U}_{\delta_n} u_n - u_n\|_{L^2(\Omega;L^2(Q;\mathbb{R}^M))} \|\nabla u_n\|_{L^2(\Omega;\mathbb{R}^{N\times M})} \leq C \left(\frac{\delta_n^2}{\varepsilon_n^2} + \frac{\delta_n}{\varepsilon_n} \right)^{\frac{1}{2}}.$$

□

3.2. **Definition and Properties of the auxiliary cell problem.** In this section we study the auxiliary cell problem which we will used in the proof of the liminf inequality (see Proposition 4.1).

Definition 3.3. Define the function $W^\eta : \mathbb{R}^M \to [0,\infty)$ as

$$W^\eta(z) := \inf_{\psi \in \mathcal{A}_\eta} \int_Q W(y, z + \psi(y))\, dy,$$

where we define the admissible set

$$\mathcal{A}_\eta := \Big\{ \psi \in L^\infty(Q;\mathbb{R}^M) \cap W^{1,2}_0(Q;\mathbb{R}^M) : \|\psi\|_{L^\infty(Q;\mathbb{R}^M)} \leq \eta, \\ \|\psi\|_{L^2(Q;\mathbb{R}^M)} \|\nabla\psi\|_{L^2(Q;\mathbb{R}^M)} \leq 5\eta^2 \Big\}.$$

We now prove the main properties of the function W^η.

Theorem 3.4 (Properties of W^η)**.** *The followings hold:*

(1) For every $z \in \mathbb{R}^M$, the infimum problem defining $W^\eta(z)$ admits a minimizer;

(2) W^η is continuous;

(3) $W^\eta(z) = 0 \iff z \in \{a, b\}$;

(4) For each $z \in \mathbb{R}^M$, $W^\eta(z)$ is increasing to $W_{\text{hom}}(z)$ as $\eta \to 0$. Moreover, W^η is converging uniformly to W_{hom} on every compact set.

Proof. **Step 1.** We prove (1). Fix $z \in \mathbb{R}^M$. Let $\{\psi_n\}_n$ be an infimizing sequence for $W^\eta(z)$. First, we note that by the definition of admissable set, we have $\sup_n \|\psi_n\|_\infty \leq \eta$. Thus, up to a subsequence (not relableled), we get that $\psi_n \overset{*}{\rightharpoonup} \psi$, for some $\psi \in L^\infty(Q; \mathbb{R}^M)$ and we also note that by definition of the convergence, we have that $\psi_n \rightharpoonup \psi$ in $L^2(Q; \mathbb{R}^M)$ as well. This is not enough to conclude by using the lower semicontinuity of the integral functional. We need to improve the convergence. To do that, we now consider two cases. *Case 1.* Suppose $\psi \neq 0$. Then, by using the constraints satisfied by each ψ_n's, we get

$$\limsup_{n\to\infty} \|\nabla \psi_n\|_{L^2(Q;\mathbb{R}^M)} \leq \limsup_{n\to\infty} \frac{5\eta^2}{\|\psi_n\|_{L^2(Q;\mathbb{R}^M)}} \leq \frac{5\eta^2}{\|\psi\|_{L^2(Q;\mathbb{R}^M)}},$$

where in the last step we used the fact that

$$\|\psi\|_{L^2(Q;\mathbb{R}^M)} \leq \liminf_{n\to\infty} \|\psi_n\|_{L^2(Q;\mathbb{R}^M)}.$$

Thus, we get that $\{\psi_n\}_n$ is bounded in $W^{1,2}(Q; \mathbb{R}^M)$. By the Rellich–Kondrachov Theorem, we get that $\psi_n \to \psi$ strongly in $L^2(Q; \mathbb{R}^M)$, and weakly in $W^{1,p}(Q; \mathbb{R}^M)$, for all $p \in (1, 2]$. In particular, by the compactness of the trace operator, we note that $\psi \in W^{1,2}_0(Q; \mathbb{R}^M)$. Thus, $\psi \in \mathcal{A}_\eta$ as we will have

$$\begin{aligned}\|\psi\|_{L^2(Q;\mathbb{R}^M)} \|\nabla\psi\|_{L^2(Q;\mathbb{R}^M)} &\leq (\liminf_{n\to\infty} \|\psi_n\|_{L^2(Q;\mathbb{R}^M)})(\liminf_{n\to\infty} \|\nabla\psi_n\|_{L^2(Q;\mathbb{R}^M)}) \\ &\leq \liminf_{n\to\infty} \|\psi_n\|_{L^2(Q;\mathbb{R}^M)} \|\nabla\psi_n\|_{L^2(Q;\mathbb{R}^M)} \leq 5\eta^2.\end{aligned}$$

Case 2. Now we consider the case that $\psi = 0$. Note that if

$$\liminf_{n\to\infty} \|\nabla\psi_n\|_{L^2(Q;\mathbb{R}^M)} < +\infty,$$

we can argue as in the previous case, since we can take a subsequence such that the sequence $\{\psi_n\}_n$ is bounded in $W^{1,2}(Q; \mathbb{R}^M)$. We now consider the case where

$$\liminf_{n\to\infty} \|\nabla\psi_n\|_{L^2(Q;\mathbb{R}^M)} = +\infty.$$

By using the constraint satisfied by the ψ_n's, we get that

$$\limsup_{n\to\infty} \|\psi_n\|_{L^2(Q;\mathbb{R}^M)} \leq \limsup_{n\to\infty} \frac{5\eta^2}{\|\nabla\psi_n\|_{L^2(Q;\mathbb{R}^M)}} \leq \frac{5\eta^2}{\liminf_{n\to\infty} \|\nabla\psi_n\|_{L^2(Q;\mathbb{R}^M)}} = 0.$$

Thus, in both cases, we achieve that $\psi_n \to \psi$ strongly in $L^2(Q; \mathbb{R}^M)$ and $\psi \in \mathcal{A}_\eta$. We then conclude by using the Dominated Convergence that

$$\lim_{n\to\infty} \int_Q W(y, z + \psi_n(y))\, dy = \int_Q W(y, z + \psi(y))\, dy.$$

Step 2. We prove (2). Let $\{z_n\}_n \subset \mathbb{R}^M$ be such that $z_n \to z$. By using step 1, for each $n \in \mathbb{N}$, there exists $\psi_n \in \mathcal{A}_\eta$ such that

$$W^\eta(z_n) = \int_Q W(y, z_n + \psi_n(y))\, dy.$$

Since the admissible set $\mathcal{A}_\eta$ does not depend on the point z_n, we get that

$$W^\eta(z) \leq \liminf_{n\to\infty} \int_Q W(y, z_n + \psi_n(y))\, dy = \liminf_{n\to\infty} W^\eta(z_n).$$

In order to establish the other inequality, let $\psi \in \mathcal{A}_\eta$ such that

$$W^\eta(z) = \int_Q W(y, z + \psi(y))\, dy.$$

Then, for each $n \in \mathbb{N}$, we get

$$W^\eta(z_n) \leq \int_Q W(y, z_n + \psi(y))\, dy \to W^\eta(z).$$

This concludes the proof of the claim.

Step 3. We prove (3). This follows directly from the fact that $W \geq 0$, that (see (W2)) $W(x, z) = 0$ if and only if $z \in \{a, b\}$, and that $\psi = 0$ on ∂Q in the sense of traces.

Step 4. We prove (4). Let $\eta_1 < \eta_2$. Then, $\mathcal{A}_{\eta_1} \subset \mathcal{A}_{\eta_2}$, and thus $W^{\eta_1}(z) \geq W^{\eta_2}(z)$ for all $z \in \mathbb{R}^M$. Let $\{\eta_n\}_n$ with $\eta_n \to 0$ as $n \to \infty$. For each $n \in \mathbb{N}$, let $\psi_n \in \mathcal{A}_{\eta_n}$ be a solution for the minimizing problem defining $W^{\eta_n}(z)$. We claim that $\psi_n \to 0$ strongly in L^2. Indeed, the sequence $\{\psi_n\}_n$ is uniformly bounded in $W^{1,2}(Q; \mathbb{R}^M)$, and since $\|\psi_n\|_{L^\infty}(Q; \mathbb{R}^M) \leq \eta_n$, we get the claim. This proves that $W^\eta(z) \to W_{\text{hom}}(z)$ as $\eta \to 0$. Since the sequence is increasing, we can apply Dini's Theorem and get that the convergence is uniform on compact sets. $\square$

3.3. **Properties of distances with degenerate metrics.** In this section we study the properties of the metrics

$$d_\eta(p, q) := \inf \left\{ \int_{-1}^{1} 2\sqrt{W^\eta(\gamma(t))}|\gamma'(t)|dt : \gamma \in \mathrm{Lip}_{\mathcal{Z}}([-1, 1]; \mathbb{R}^M),\, \gamma(-1) = p,\, \gamma(1) = q \right\}.$$

In particular, we are interested in their behavior as $\eta \to 0$. We first state some basic properties that are easy to establish.

Lemma 3.5. *For $p, q \in \mathbb{R}^M$, define*

$$d_0(p, q) := \sup_{\eta > 0} d_\eta(p, q).$$

Then

(1) d_0 is a metric on $\mathbb{R}^M$;

(2) For each $p, q \in \mathbb{R}^M$, it holds $\lim_{\eta\to 0} d_\eta(p, q) = d_0(p, q)$.

The existence of minimizing geodesics has already been established in [34] for each η (see Proposition 2.5, Item 3). Therefore, for each $\eta > 0$, we have $\gamma_\eta \in \mathrm{Lip}_{\mathcal{Z}}([0,1];\mathbb{R}^M)$ with $\gamma_\eta(-1) = a$, $\gamma_\eta(1) = b$ such that

$$\sigma_\eta := d_\eta(a,b) = \int_{-1}^{1} 2\sqrt{W^\eta(\gamma_\eta)}|\gamma_\eta'|dt.$$

We now fix want to investigate the behavior of minimizing curves γ_η.

Lemma 3.6. *Let $\{\eta_n\}_n$ be an infinitesimal sequence, and let $\{\gamma_{\eta_n}\}_n$ be a sequence of geodesics for $d_{\eta_n}(a,b)$. Then, (up to a subsequence, not relabeled) there exists a curve $\gamma_0 \in \mathrm{Lip}_{\mathcal{Z}}([-1,1];\mathbb{R}^M)$ with $\gamma_0(-1) = a$, $\gamma_0(1) = b$ such that*

$$\lim_{n\to\infty} \sup_{t\in[-1,1]} d_0(\gamma_{\eta_n}(t), \gamma_0(t)) = 0.$$

Proof. **Step 1.** We want to apply the Ascoli-Arzelà Theorem. First of all, note that the at least quadratic growth of W at infinity, together with the fact that each $\psi \in \mathcal{A}_\eta$ is bounded in L^∞, yields that W^{η_n} grows at least quadratically at infinity. Then, it is possible to find $M > 0$ such that

$$\{\gamma_{\eta_n}(t) : \ t \in [0,1], \ \ \eta > 0\} \subset \overline{B(0,M)},$$

for all $n \in \mathbb{N}$, where $B(0,M)$ is the Euclidean ball.

By using the upper bound of d_0 by d_{hom} and choosing the straight line path between the point a and the termination point (denoted by $L_{a,p}$), we can compute

$$\sup_{t\in[-1,1]} d_0(a,\gamma_{\eta_n}(t)) \leq \sup_{t\in[-1,1]} d_{\mathrm{hom}}(a,\gamma_{\eta_n}(t)) \leq \sup_{t\in[-1,1]} 2|\gamma_{\eta_n}(t) - a| \int_{-1}^{1} \sqrt{W_{\mathrm{hom}}(L_{a,\gamma_{\eta_n}(t)})}ds$$

using the continuity of $\sqrt{W_{\mathrm{hom}}}$ and the boundedness of $\gamma_{\eta_n}(t)$ in the Euclidean ball $B(0,M)$, we can establish the uniform boundedness in the d_0 metric

$$\{\gamma_{\eta_n}(t) : \ t \in [0,1], \ \ \eta > 0\} \subset \overline{B_{d_0}(a, 4\|\sqrt{W_{\mathrm{hom}}}\|_{L^\infty(B(0,M))}(|M| + |a|))}.$$

We now prove the equicontinuity of the sequence with respect to the metric d_0. We we show that the sequence is equi-Lipschitz. We use the idea from [34, Proof of Theroem 2.6] of rescaling by the degenerate arclength. For any $n \in \mathbb{N}$, it is possible to rescale the curve γ_η to a curve (that with an abuse of notation we still denote by) $\gamma_{\eta_n} : [-1,1]$ in such a way that

$$2\sqrt{W^{\eta_n}(\gamma_{\eta_n}(t))}|\gamma_{\eta_n}'(t)| \equiv \frac{\sigma_{\eta_n}}{2},$$

for all $t \in [-1,1]$. Let $t_2 > t_1$. Then,

$$d_{\eta_n}(\gamma_{\eta_n}(t_1), \gamma_{\eta_n}(t_2)) = \int_{t_1}^{t_2} 2\sqrt{W^{\eta_n}(\gamma_{\eta_n}(t))}|\gamma_{\eta_n}'(t)|\, dt = \frac{\sigma_{\eta_n}}{2}(t_2 - t_1) \leq \frac{\sigma_0}{2}(t_2 - t_1).$$

Therefore, we get that

$$d_0(\gamma_{\eta_n}(t_1), \gamma_{\eta_n}(t_2)) \leq \frac{\sigma_0}{2}(t_2 - t_1) + \|d_{\eta_n} - d_0\|_\infty,$$

where the last norm is the uniform norm in the space $\overline{B(0,M)} \times \overline{B(0,M)}$. Note that, by definition of d_0, and by using Dini's Theorem on the compact space $\overline{B(0,M)} \times \overline{B(0,M)}$, we get that

$$\lim_{n\to\infty} \|d_{\eta_n} - d_0\|_\infty = 0.$$

Fix $\varepsilon > 0$. Then, there exists $\bar{n} \in \mathbb{N}$ such that

$$\|d_{\eta_n} - d_0\|_\infty < \frac{\varepsilon}{2}.$$

Let

$$\delta_0 := \frac{\varepsilon}{\sigma_0}.$$

Then, for every $n \geq \bar{n}$, it holds

$$d_0(\gamma_{\eta_n}(t_1), \gamma_{\eta_n}(t_2)) \leq \varepsilon,$$

whenever $|t_2 - t_1| < \delta_0$. For each $n = 1, \dots, \bar{n}$, by using the uniform continuity of γ_{η_n}, let $\delta_n > 0$ be such that

$$d_0(\gamma_{\eta_n}(t_1), \gamma_{\eta_n}(t_2)) \leq \varepsilon$$

whenever $|t_2 - t_1| < \delta_n$. Define

$$\delta := \min\{\delta_0, \delta_1, \dots, \delta_{\bar{n}}\}.$$

This proves that the sequence $\{\gamma_{\eta_n}\}_{n\in\mathbb{N}}$ is equicontinuous. Thus, we get the existence of a subsequence, and of a curve $\gamma_0 : [-1, 1] \to \mathbb{R}^M$ to which the subsequence converges uniformly with respect to the metric d_0.

Step 2. We now prove that $\gamma_0 \in \mathrm{Lip}_{\mathcal{Z}}([-1, 1]; \mathbb{R}^M)$. This follows directly from the fact that, given any $r > 0$, there exists $m > 0$ such that

$$\inf \left\{ W^{\eta_n}(z) \,:\, z \in \mathbb{R}^M \setminus (B(a, r) \cup B(b, r)) \right\} \geq m,$$

for all $n \in \mathbb{N}$. Indeed, this follows from the uniform convergence of W^{η_n} to W_{hom}, together with the fact that W_{hom} only vanishes on a and b. □

Now we are ready to prove the main result of this section.

Proposition 3.7. *It holds*

$$d_0(a, b) = \lim_{\eta\to0} \int_{-1}^{1} 2\sqrt{W^\eta(\gamma_0)}|\gamma_0'|dt = \sup_{\eta>0} \int_{-1}^{1} 2\sqrt{W^\eta(\gamma_0)}|\gamma_0'|dt.$$

Proof. It is easy to see by using the definition that

$$d_\eta(a, b) \leq \int_{-1}^{1} 2\sqrt{W^\eta(\gamma_0)}|\gamma_0'|dt.$$

Thus taking the limit as $\eta \to 0$, we get

$$d_0(a, b) \leq \lim_{\eta\to0} \int_{-1}^{1} 2\sqrt{W^\eta(\gamma_0)}|\gamma_0'|dt.$$

Now we prove the other inequality. By using Proposition 2.5 Item 2 and the fact that $\gamma_0 \in \mathrm{Lip}_{\mathcal{Z}}([-1, 1]; \mathbb{R}^M)$, we get

$$\int_{-1}^{1} 2\sqrt{W^\eta(\gamma_0)}|\gamma_0'|dt = L_\eta(\gamma_0).$$

Fix an arbitrary finite partition $\{t_k\}_{k=1}^m$ of $[-1.1]$. We note that we can use the triangle inequality and the definition of d_0 to get

$$\begin{aligned} d_\eta(\gamma_0(t_k), \gamma_0(t_{k+1})) &\leq d_\eta(\gamma_\eta(t_k), \gamma_0(t_k)) + d_\eta(\gamma_\eta(t_k), \gamma_\eta(t_{k+1})) + d_\eta(\gamma_0(t_k), \gamma_\eta(t_{k+1})) \\ &\leq d_0(\gamma_\eta(t_k), \gamma_0(t_k)) + d_\eta(\gamma_\eta(t_k), \gamma_\eta(t_{k+1})) + d_0(\gamma_0(t_k), \gamma_\eta(t_{k+1})). \end{aligned}$$

Fix $j \in \mathbb{N}$. Then, using the definition of uniform convergence we can find $\eta_0(j,m)$ such that for all $\eta < \eta_0$ we have

$$\sup_{t\in[-1,1]} d_0(\gamma_\eta(t), \gamma_0(t)) \leq \frac{1}{jm}.$$

Thus, we can bound the total sum over the partitions using the uniform convergence estimate for all $\eta < \eta_0$ by:

$$\sum_{k=1}^{m} d_\eta(\gamma_0(t_k), \gamma_0(t_{k+1})) \leq \sum_{k=1}^{m} d_\eta(\gamma_\eta(t_k), \gamma_\eta(t_{k+1})) + \frac{1}{j} = L_\eta(\gamma_\eta) + \frac{1}{j} = d_\eta(a,b) + \frac{1}{j}.$$

Thus, by taking a supremum over all possible finite partitions, we get that

$$\int_{-1}^{1} 2\sqrt{W^\eta(\gamma_0)}|\gamma_0'|dt = L_\eta(\gamma_0) \leq d_\eta(a,b) + \frac{1}{j}$$

By taking the limit as $\eta \to 0$ and then as $j \to \infty$, we get the required result. □

4. Liminf Inequality

The goal of this section is to prove the following result.

Proposition 4.1. *Let $\{u_n\}_n \subset W^{1,2}(\Omega;\mathbb{R}^M)$ be such that $u_n \to u \in BV(\Omega;\{a,b\})$ strongly in $L^2(\Omega;\mathbb{R}^M)$. Then,*

$$\liminf_{n\to\infty} \mathcal{E}_n(u_n) \geq \mathcal{E}_\infty(u).$$

Note that this proposition is weaker than the liminf inequality, since we are assuming the limiting function to be in $BV(\Omega;\{a,b\})$, and not just in $L^2(\Omega;\{a,b\})$. We will prove in the next section that this is sufficient. The reason why we are first proving the above result is that, in the proof of the compactness (see Theorem 1.4), we will use computations that are the core of the idea to prove the above result. Thus, we prefer to present them first here.

We will split the proof up into different steps.

Step 0: Reduction In this step, we show that it suffices to prove Proposition 4.1 for u_n uniformly bounded in L^∞, and W with strictly quadratic growth at infinity. Let $M > R_1$, where $R_1 > 0$ is the constant given in (W3). Let $\psi_M : [0,+\infty) \to [0,1]$ be a smooth function such that

$$\psi_M \equiv 1 \text{ on } [0,M], \qquad \psi_M \equiv 0 \text{ for } t \geq 2M.$$

Define

$$\widetilde{W}^M(x,z) := \psi_M(|z|)W(x,z) + (1-\psi(|z|))\frac{|z|^2}{R_1},$$

and

$$\widetilde{\mathcal{E}}_n(v) := \int_\Omega \frac{1}{\varepsilon_n}\left[\widetilde{W}^M\left(\frac{x}{\delta_n}, v(x)\right) + \varepsilon_n|\nabla v(x)|^2\right] dx.$$

We claim the following. Assume that for all $v \in BV(\Omega;\{a,b\})$ it holds

$$\mathcal{E}_\infty(v) \leq \liminf_{n\to\infty} \widetilde{\mathcal{E}}_n(v_n),$$

for all $\{v_n\}_n \subset W^{1,2}(\Omega;\mathbb{R}^M)$ with $\|v_n\|_{L^\infty} \leq 2M$, such that $v_n \to v$ strongly in $L^2(\Omega;\mathbb{R}^M)$. Then, Proposition 4.1 holds.

Indeed, let $\{u_n\}_n \subset W^{1,2}(\Omega;\mathbb{R}^M)$ be such that $u_n \to u \in BV(\Omega;\{a,b\})$ strongly in $L^2(\Omega;\mathbb{R}^M)$. Let

$$v_n := \mathcal{T}_{2M} u_n,$$

where $\mathcal{T}_{2M} u_n$ is the truncation of v (see Definition 2.3). First of all, we claim that

$$\limsup_{n\to\infty} \|u_n - v_n\|_{L^2(\Omega;\mathbb{R}^M)} = 0. \tag{4.1}$$

Indeed, let

$$\Delta_n := \{x \in \Omega \ : \ |u_n(x)| > 2M\}.$$

By using Chebyshev's inequality, we get

$$\mathcal{L}^N(\Delta_n) \leq \frac{1}{4M^2}\int_{\Delta_n} |u_n|^2 \, dx \leq \frac{1}{4M^2}\int_{\Omega} W\left(\frac{x}{\delta_n}, u_n\right) \, dx \leq C_M \varepsilon_n, \tag{4.2}$$

where in the second inequality we used the fact that $M > R_1$ together with (W3). Then, by using the triangle inequality and convexity, we obtain

$$\begin{aligned}
\|u_n - v_n\|^2_{L^2(\Omega;\mathbb{R}^M)} &= \int_{\Delta_n} |u_n - v_n|^2 \, dx \\
&\leq C_M \left(\int_{\Delta_n} |u_n|^2 \, dx + \mathcal{L}^N(\Delta_n) \right) \\
&\leq C_M \varepsilon_n,
\end{aligned}$$

where in the last step we used item (W3) and (4.2). This proves (4.1). In particular, we get that $v_n \to u$ strongly in $L^2(\Omega;\mathbb{R}^M)$.

Therefore, we get

$$\mathcal{E}_\infty(v) \leq \liminf_{n\to\infty} \widetilde{\mathcal{E}}(v_n).$$

To conclude, we claim that

$$\widetilde{\mathcal{E}}(v_n) \leq \mathcal{E}(u_n),$$

for all $n \in \mathbb{N}$. Indeed, note that $\widetilde{W}_M$ satisfies (W1), (W2), and (W3). Thus, by (W3), we get

$$\widetilde{W}^M(x, v_n(x)) \leq \widetilde{W}^M(x, u_n(x)) \leq W(x, u_n).$$

for almost all $x \in \Omega$, and all $z \in \mathbb{R}^M$.

This, together with Lemma 2.4, yields

$$\int_\Omega \frac{1}{\varepsilon_n}\left[\widetilde{W}^M\left(\frac{x}{\delta_n}, v_n(x)\right) + \varepsilon_n |\nabla v_n(x)|^2\right] \, dx \leq \mathcal{E}_n(u_n),$$

and thus the desired conclusion.

In the rest of this section, we will therefore assume that

$$\sup_{n\in\mathbb{N}} \|u_n\|_{L^\infty(\Omega;\mathbb{R}^M)} \leq M, \tag{4.3}$$

for some $M > R_1$. Moreover, by passing to a subsequence, without loss of generality, we will also assume that

$$\liminf_{n\to\infty} \mathcal{E}_n(u_n) = \lim_{n\to\infty} \mathcal{E}_n(u_n) < \infty.$$

Step 1: Slicing For each $n \in \mathbb{N}$, and $\eta > 0$, let

$$f_n^\eta(x,y) := \mathcal{T}_\eta(\mathcal{U}_{\delta_n} u_n(x,y) - u_n(x)),$$

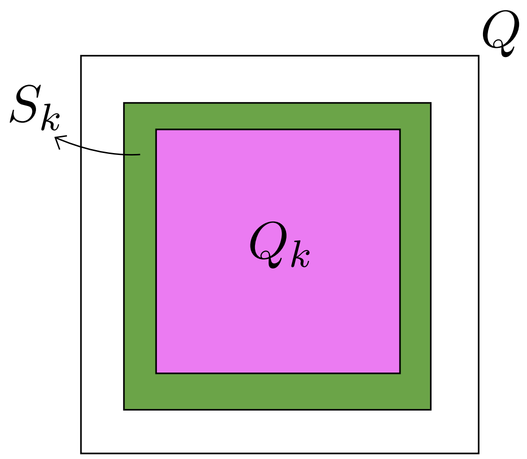

FIGURE 1. The construction of the sets Q_k, and S_k, where the cut-off function ψ_k equals one, and transition from one to zero, respectively.

where the truncation operator $\mathcal{T}_\eta$ is with respect to the variable y. We want use a De Giorgi's slicing type of argument to modify f_n^η to make it vanish on $\Omega \times \partial Q$.

For $k \in \mathbb{N} \setminus \{0\}$, let (see Figure 1)

$$S_k := \left\{ y \in Q \ : \ \frac{1}{k} < \operatorname{dist}(y, \partial Q) \leq \frac{2}{k} \right\},$$

and

$$Q_k := \left\{ y \in Q \ : \ \frac{2}{k} < \operatorname{dist}(y, \partial Q) \right\}.$$

Let $\psi_k : Q \to [0,1]$ be a smooth function with $0 \leq \psi \leq 1$ such that

$$\psi_k \equiv 1 \text{ on } Q_k, \qquad \psi_k \equiv 0 \text{ on } Q \setminus (Q_k \cup S_k), \qquad \|\nabla \psi\|_{L^\infty(Q)} \leq k.$$

Define

$$v_{n,k}^\eta(x,y) = \psi_k(y) f_n^\eta(x,y).$$

We claim that the following holds.

Theorem 4.2. *There exists a sequence $\{k_n^\eta(x)\}_n$ with $x \mapsto k_n^\eta(x)$ measurable,*

$$\lim_{n\to\infty} k_n^\eta(x) = +\infty$$

for almost every $x \in \Omega$, and

$$\limsup_{n\to\infty} \int_\Omega \frac{1}{k_n(x)\varepsilon_n} \, dx = 0, \tag{4.4}$$

that satisfies the following properties. Denoting $v_n^\eta := v_{n,k_n(x)}^\eta$, it holds

$$\limsup_{n\to\infty} \mathcal{E}_n(u_n) \geq \limsup_{n\to\infty} \int_\Omega 2 \left[\int_Q W(y, u_n + v_n^\eta) \, dy \right]^{\frac{1}{2}} |\nabla u_n| \, dx, \tag{4.5}$$

and, for almost every $x \in \Omega$, we have the estimate

$$\|v_n^\eta(x,\cdot)\|_{L^2(Q;\mathbb{R}^M)} \|\nabla_y v_n^\eta(x,\cdot)\|_{L^2(Q;\mathbb{R}^M)} \leq 4\eta^2 + \eta \|\nabla_y \mathcal{U}_{\delta_n} u_n(x,\cdot)\|_{L^2(Q;\mathbb{R}^{N\times M})}. \tag{4.6}$$

The proof of the above theorem requires some preliminary results. First of all, we claim that, for almost all $x \in \Omega$, it holds that

$$\|v^\eta_{n,k}(x,\cdot)\|_{L^\infty(Q;\mathbb{R}^M)} \leq \eta, \tag{4.7}$$

for all $n \in \mathbb{N}$, and that

$$\lim_{n\to\infty} \|v^\eta_{n,k}(x,\cdot)\|_{L^2(Q;\mathbb{R}^M)} \leq \lim_{n\to\infty} \|f^\eta_n(x,\cdot)\|_{L^2(Q;\mathbb{R}^M)} = 0. \tag{4.8}$$

Indeed, the first estimate is direct from the definition of the function $v^\eta_{n,k}$ together with the fact that the cut-off function is bounded above by one. Moreover, (4.8) follows from (3.4) after taking a subsequence. We will work with this subsequence without relabeling.

Now, we estimate

$$\begin{aligned}
&\int_\Omega 2\left[\int_Q W(y, u_n + v^\eta_{n,k})\, dy\right]^{\frac{1}{2}} |\nabla u_n|\, dx \\
&\qquad \leq \int_\Omega 2\left[\int_Q W(y, u_n + f^\eta_n)\, dy\right]^{\frac{1}{2}} |\nabla u_n|\, dx \\
&\qquad\qquad + \int_\Omega 2\left[\int_{Q\setminus Q_{k(x)}} W(y, u_n)\, dy\right]^{\frac{1}{2}} |\nabla u_n|\, dx \\
&\qquad\qquad + \int_\Omega 2\left[\int_{S_{k(x)}} W(y, u_n + v^\eta_{n,k}(x,y))\, dy\right]^{\frac{1}{2}} |\nabla u_n|\, dx \\
&\qquad =: I_n + II_n + III_n
\end{aligned} \tag{4.9}$$

We want to estimate the three integrals separately. We do this in separate lemmata.

Lemma 4.3 (Estimate for I_n)**.** *It holds*

$$\begin{aligned}
\limsup_{n\to\infty} &\int_\Omega 2\left[\int_Q W(y, u_n + f^\eta_n)\, dy\right]^{\frac{1}{2}} |\nabla u_n|\, dx \\
&\leq \limsup_{n\to\infty} \int_\Omega 2\left[\int_Q W(y, \mathcal{U}_{\delta_n} u_n)\, dy\right]^{\frac{1}{2}} |\nabla u_n|\, dx.
\end{aligned}$$

Proof. Write

$$\begin{aligned}
&\int_\Omega 2\left[\int_Q W(y, u_n + \mathcal{T}_\eta(\mathcal{U}_{\delta_n} u_n - u_n))\, dy\right]^{\frac{1}{2}} |\nabla u_n|\, dx \\
&\qquad \leq \int_\Omega 2\left[\int_Q W(y, \mathcal{U}_{\delta_n} u_n)\, dy\right]^{\frac{1}{2}} |\nabla u_n|\, dx \\
&\qquad\qquad + \int_\Omega 2\left[\int_{Q^\eta_n(x)} W(y, u_n + \mathcal{T}_\eta(\mathcal{U}_{\delta_n} u_n - u_n))\, dy\right]^{\frac{1}{2}} |\nabla u_n|\, dx,
\end{aligned}$$

where

$$Q^\eta_n(x) := \{y \in Q \;:\; |\mathcal{U}_{\delta_n} u_n(x,y) - u_n(x)| > \eta\}.$$

Note that, by Chebyshev inequality,

$$\mathcal{L}^N(Q^\eta_n(x)) \leq \frac{1}{\eta^2}\int_Q |\mathcal{U}_{\delta_n} u_n(x,y) - u_n(x)|^2\, dy = \frac{1}{\eta^2}\|\mathcal{U}_{\delta_n} u_n(x,\cdot) - u_n(x)\|^2_{L^2(Q;\mathbb{R}^M)}.$$

By using (W3), and (4.3), we get

$$\begin{aligned}
\int_\Omega \left[\int_{Q_n^\eta(x)} W(y, u_n + \mathcal{T}_\eta(\mathcal{U}_{\delta_n} u_n - u_n))\, dy \right]^{\frac{1}{2}} |\nabla u_n|\, dx \\
&\leq C_{M,\eta} \int_\Omega \mathcal{L}^N(Q_n^\eta(x))^{\frac{1}{2}} |\nabla u_n(x)|\, dx \\
&\leq C_{M,\eta} \|\mathcal{L}^N(Q_n^\eta(x))^{\frac{1}{2}}\|_{L^2(\Omega\, \mathbb{R}^M)} \|\nabla u_n\|_{L^2(\Omega\, \mathbb{R}^{N\times M})} \\
&= C_{M,\eta} \|\mathcal{U}_{\delta_n} u_n - u_n\|_{L^2(\Omega; L^2(Q;\mathbb{R}^M))} \|\nabla u_n\|_{L^2(\Omega;\mathbb{R}^{N\times M})} \\
&\leq C_{M,\eta} \left(\frac{\delta_n^2}{\varepsilon_n^2} + \frac{\delta_n}{\varepsilon_n} \right)^{\frac{1}{2}},
\end{aligned}$$

where we used Hölder and estimate (3.5). Since $\delta_n \ll \varepsilon_n$, this gives

$$\begin{aligned}
\limsup_{n\to\infty} \int_\Omega 2 \left[\int_Q W(y, u_n + \mathcal{T}_\eta(\mathcal{U}_{\delta_n} u_n - u_n))\, dy \right]^{\frac{1}{2}} |\nabla u_n|\, dx \\
\leq \limsup_{n\to\infty} \int_\Omega 2 \left[\int_Q W(y, \mathcal{U}_{\delta_n} u_n)\, dy \right]^{\frac{1}{2}} |\nabla u_n|\, dx.
\end{aligned}$$

□

Now, we estimate II_n.

Lemma 4.4 (Estimate for II_n)**.** *Assume*

$$\limsup_{n\to\infty} \int_\Omega \frac{1}{k_n(x)\varepsilon_n}\, dx = 0.$$

Then,

$$\lim_{n\to\infty} II_n = 0.$$

Proof. By the growth of W (see (W3)), we get

$$\begin{aligned}
\int_\Omega 2 \left[\int_{Q\setminus Q_{k_n(x)}} W(y, u_n)\, dy \right]^{\frac{1}{2}} |\nabla u_n|\, dx &\leq C \int_\Omega \left[\int_{Q\setminus Q_{k_n(x)}} (1 + |u_n|^2)\, dy \right]^{\frac{1}{2}} |\nabla u_n|\, dx \\
&\leq C \int_\Omega \mathcal{L}^N(Q \setminus Q_{k_n(x)})^{\frac{1}{2}} |\nabla u_n|\, dx \\
&\leq C \|\mathcal{L}^N(Q \setminus Q_{k_n(x)})^{\frac{1}{2}}\|_{L^2(\Omega)} \|\nabla u_n\|_{L^2(\Omega)},
\end{aligned}$$

where in the previous to last step we used the fact that $\|u_n\|_\infty \leq M$ for all $n \in \mathbb{N}$ (recall (4.3)), while last step follows from Hölder inequality. Now, by using estimate (3.1) together with the fact that

$$\mathcal{L}^N(Q \setminus Q_{k_n(x)}) \leq \frac{C}{k_n(x)},$$

we get

$$II_n \leq C \left[\int_\Omega \frac{1}{k_n(x)\varepsilon_n}\, dx \right]^{\frac{1}{2}}.$$

By using the assumption, we conclude. □

We finally estimate III_n by using a similar strategy.

Lemma 4.5 (Estimate for III_n)**.** *Assume*

$$\limsup_{n\to\infty} \int_\Omega \frac{1}{k_n(x)\varepsilon_n}\, dx = 0.$$

Then,

$$\lim_{n\to\infty} III_n = 0.$$

Proof. By the growth of W (see (W3)), we get

$$\int_\Omega 2\left[\int_{S_{k(x)}} W(y, u_n + v^\eta_{n,k}(x,y))\, dy\right]^{\frac{1}{2}} |\nabla u_n|\, dx$$
$$\leq C\int_\Omega \left[\int_{S_{k(x)}} 1 + |u_n|^2 + |f^\eta_n|^2\, dy\right]^{\frac{1}{2}} |\nabla u_n|\, dx,$$

where we used the fact that $\psi_{k,n} \leq 1$. Now, by using (4.3) together with $|f^\eta_n| \leq \eta$, we get

$$III_n \leq C_{M,\eta}\|\mathcal{L}^N(S_{k(x)})^{\frac{1}{2}}\|_{L^2(\Omega)}\|\nabla u_n\|_{L^2(\Omega)}.$$

By using estimate (3.1), the fact that

$$\mathcal{L}^N(S_{k(x)}) \leq \frac{C}{k_n(x)},$$

and the assumption, we conclude. □

We are now in position to prove the main theorem of this step.

Proof of Theorem 4.2. **Step 1.** Define

$$k_n(x) := \left\lfloor \frac{\eta^2}{\|f^\eta_n(x,\cdot)\|^2_{L^2(Q;\mathbb{R}^M)}} \right\rfloor.$$

First of all, k_n is a Lebesgue measurable function of x as it is the composition of an upper semicontinuous function (hence Borel measurable) and a Lebesgue measurable function. Furthermore, by (4.8) we have that the denominator converges to zero, we have that $k_n \to \infty$ as $n \to \infty$. Moreover, in view of (3.4), we have

$$\int_\Omega \frac{1}{\varepsilon_n}\|f^\eta_n(x,\cdot)\|^2_{L^2(Q;\mathbb{R}^M)}\, dx \leq C\left(\frac{\delta_n^2}{\varepsilon_n^2} + \frac{\delta_n}{\varepsilon_n}\right)$$

Thus, using that $\delta_n \ll \varepsilon_n$, we achieve

$$\limsup_{n\to\infty} \int_\Omega \frac{1}{k_n(x)\varepsilon_n}\, dx = 0.$$

Step 2. We now prove the energy estimate . From (4.9), together with Lemma 4.3, 4.4, and 4.5, that we can apply, thanks to (4.4), we get the desired estimate.

Step 3. Finally, we establish the bound (4.6). Note that

$$|\nabla_y v^\eta_{k,n}|^2 = |\nabla_y \psi_k f^\eta_n + \psi_k \nabla_y f^\eta_n|^2 \leq 2(k^2|f^\eta_n|^2 + |\nabla_y f^\eta_n|^2),$$

and thus that

$$\|\nabla_y v^\eta_{k,n}(x,\cdot)\|_{L^2(Q;\mathbb{R}^M)} \leq 2\left[k\|f^\eta_n(x,\cdot)\|_{L^2(Q;\mathbb{R}^M)} + \|\nabla_y f^\eta_n(x,\cdot)\|_{L^2(Q;\mathbb{R}^M)}\right].$$

Since

$$\|v^\eta_{k,n}\|_{L^2(Q;\mathbb{R}^M)} \leq \|f^\eta_n\|_{L^2(Q;\mathbb{R}^M)},$$

we obtain

$$\begin{aligned}
&\|v^\eta_{k,n}(x,\cdot)\|_{L^2(Q;\mathbb{R}^M)}\|\nabla_y v^\eta_{k,n}(x,\cdot)\|_{L^2(Q;\mathbb{R}^M)} \\
&\qquad \leq 2\left[k\|f^\eta_n(x,\cdot)\|^2_{L^2(Q;\mathbb{R}^M)} + \|f^\eta_n(x,\cdot)\|_{L^2(Q;\mathbb{R}^M)}\|\nabla_y f^\eta_n(x,\cdot)\|_{L^2(Q;\mathbb{R}^M)}\right] \\
&\qquad \leq 2\left[2\eta^2 + \eta\|\nabla_y \mathcal{U}_{\delta_n} u_n(x,\cdot)\|_{L^2(Q;\mathbb{R}^M)}\right],
\end{aligned}$$

where in the last step we used the fact that the gradient of the truncation is bounded pointwise by the gradient of the original function. This concludes the proof of the theorem. □

Step 2: Passing to Limit By using the result of step 1, we get that, for all $\eta > 0$, it holds

$$\liminf_{n\to\infty} \mathcal{E}_n(u_n) \geq \liminf_{n\to\infty} \int_\Omega 2\left[\int_Q W(y, u_n + v^\eta_n)\, dy\right]^{\frac{1}{2}} |\nabla u_n|\, dx.$$

We now want to pass to the limit in the above inequality as $\eta \to 0$. For, we would like to use the function W^η. The problem is that $v^\eta_n(x,\cdot)$ might not satisfy, for all $x \in \Omega$, the require bound to be an admissible competitor for the minimization problem defining W^η. Thus, we reason as follows. Define

$$\Omega^\eta_n = \{x \in \Omega : \ \|\nabla_y \mathcal{U}_{\delta_n} u_n(x,\cdot)\|_{L^2(Q;\mathbb{R}^M)} \leq \eta\}.$$

Then, for each $x \in \Omega^\eta_n$, we have that $v^\eta_n(x,\cdot) \in \mathcal{A}_\eta$ (see Definition 3.3). We can estimate

$$\begin{aligned}
\liminf_{n\to\infty} \int_\Omega 2\left[\int_Q W(y, u_n + v^\eta_n)\, dy\right]^{\frac{1}{2}} |\nabla u_n|\, dx &\geq \liminf_{n\to\infty} \int_{\Omega^\eta_n} 2\sqrt{W^\eta(u_n)}|\nabla u_n|\, dx \\
&\geq \liminf_{n\to\infty} \int_\Omega 2\sqrt{W^\eta(u_n)}|\nabla u_n|\, dx - \limsup_{n\to\infty} \int_{\Omega\setminus\Omega^\eta_n} 2\sqrt{W^\eta(u_n)}|\nabla u_n|\, dx.
\end{aligned}$$

We claim that

$$\limsup_{n\to\infty} \int_{\Omega\setminus\Omega^\eta_n} 2\sqrt{W^\eta(u_n)}|\nabla u_n|\, dx = 0.$$

Indeed, by Chebyshev's inequality, we have that

$$\mathcal{L}^N(\Omega \setminus \Omega^\eta_n) \leq \frac{1}{\eta^2} \int_\Omega \|\nabla_y \mathcal{U}_{\delta_n} u_n(x,\cdot)\|^2_{L^2(Q;\mathbb{R}^M)}\, dx,$$

and thus, by using (3.3), we get

$$\mathcal{L}^N(\Omega \setminus \Omega^\eta_n) \leq \frac{\delta_n^2}{\eta^2 \varepsilon_n}. \tag{4.10}$$

By using the upper bound on u_n (see (4.3)) we obtain

$$\int_{\Omega\setminus\Omega^\eta_n} 2\sqrt{W^\eta(u_n)}|\nabla u_n|\, dx \leq C_{M,\eta} \int_{\Omega\setminus\Omega^\eta_n} |\nabla u_n|\, dx.$$

Now applying Hölder, and (4.10) we infer that

$$\int_{\Omega\setminus\Omega^\eta_n} 2\sqrt{W^\eta(u_n)}|\nabla u_n|\, dx \leq C_{M,\eta}\mathcal{L}^N(\Omega \setminus \Omega^\eta_n)^{\frac{1}{2}}\|\nabla u_n\|_{L^2(Q;\mathbb{R}^M)} \leq C_{M,\eta}\frac{\delta_n}{\varepsilon_n},$$

that proves the claim using that $\delta_n \ll \varepsilon_n$.

Therefore, for all $\eta > 0$, we have that

$$\lim_{n\to\infty} \mathcal{E}_n(u_n) \geq \liminf_{n\to\infty} \int_\Omega 2\sqrt{W^\eta(u_n)}|\nabla u_n|\, dx.$$

We specifically note that up to this point, we have not used the convergence of u_n in any way. Now we do so, thanks to classical results on the Modica-Mortola functional (see [19, Theorem 3.4]), we get that

$$\lim_{n\to\infty} \mathcal{E}_n(u_n) \geq \sigma_\eta \mathrm{Per}(\{u = a\}), \tag{4.11}$$

where

$$\sigma_\eta := \min\left\{ \int_{-1}^1 2\sqrt{W^\eta(\gamma)}|\gamma'|dt \,:\, \gamma \in \mathrm{Lip}_{\mathcal{Z}}([-1,1];\mathbb{R}^M), \gamma(-1) = a,\, \gamma(1) = b \right\}.$$

Step 4: Concluding the Arguments To conclude the argument, we send $\eta \to 0$ in (4.11). Recall that

$$\sigma_{\mathrm{hom}} := \min\left\{ \int_{-1}^1 2\sqrt{W_{\mathrm{hom}}(\gamma)}|\gamma'|dt \,:\, \gamma \in \mathrm{Lip}_{\mathcal{Z}}([-1,1];\mathbb{R}^M), \gamma(-1) = a,\, \gamma(1) = b \right\}.$$

We claim the following.

Proposition 4.6. *Let*

$$\sigma_0 := \lim_{\eta\to 0} \sigma_\eta = \sup_{\eta>0} \sigma_\eta.$$

Then, $\sigma_0 = \sigma_{\mathrm{hom}}$.

Proof. First of all, we note that σ_η is increasing, and thus the limit exists.

Now, by Theorem 3.4, for every $\eta > 0$, $\sigma_\eta \leq \sigma_{\mathrm{hom}}$. This establishes the inequality $\sigma_0 \leq \sigma_{\mathrm{hom}}$.

To prove the other inequality, we reason as follows. First of all, we note that Lemma 3.5 gives us that σ_0 is the distance between a and b in a certain metric d_0, and that there exists $\gamma_0 \in C^0([-1,1];\mathbb{R}^M)$ such that

$$\sigma_0 = \sup_{\eta>0} \int_{-1}^1 2\sqrt{W^\eta(\gamma_0)}|\gamma_0'|dt. \tag{4.12}$$

Without loss of generality, due to parameterization invariance, we can assume that the $\gamma_0(t) = a$ if and only if $t = -1$, and that $\gamma_0(t) = b$ if and only if $t = 1$. For $j \in \mathbb{N} \setminus \{0\}$, define

$$T_j^a := \{\, t \in [-1,1] \,:\, \gamma_0(t) \notin \overline{B(a, 1/j)} \,\},$$
$$T_j^b := \{\, t \in [-1,1] \,:\, \gamma_0(t) \notin \overline{B(b, 1/j)} \,\},$$

and

$$T_j := [-1,1] \setminus (T_j^a \cup T_j^b).$$

Let

$$L_j := \int_{T^j} |\gamma_0'(t)|\, dt < \infty.$$

By Theorem 3.4, we have uniform convergence of W^η ot W_{hom} on compact sets. Thus, let $(\eta_j)_j$ be a decreasing sequence such that

$$\|\sqrt{W^\eta} - \sqrt{W_{\mathrm{hom}}}\|_{C^0(\overline{K\setminus T^j})} \leq \frac{1}{2jL_j}.$$

for all $\eta < \eta_j$, where $K \subset \mathbb{R}^M$ is a compact set such that $\gamma_0(t) \in K$ for all $t \in [-1,1]$. We are now in position to conclude the the proof

$$\begin{aligned}\sigma_0 &\geq \int_{-1}^{1} 2\sqrt{W^\eta(\gamma_0)}|\gamma_0'|dt \\ &\geq \int_{T^j} 2\sqrt{W^\eta(\gamma_0)}|\gamma_0'|dt \\ &\geq \int_{T^j} 2\sqrt{W_{\mathrm{hom}}(\gamma_0)}|\gamma_0'|dt - 2\int_{T^j}\left|\sqrt{W^\eta(\gamma_0)} - \sqrt{W_{\mathrm{hom}}(\gamma_0)}\right||\gamma_0'|dt \\ &\geq \int_{T^j} 2\sqrt{W_{\mathrm{hom}}(\gamma_0)}|\gamma_0'|dt - \frac{1}{j}\end{aligned}$$

Taking $j \to \infty$ and using Monotone Convergence Theorem together with (4.12), we conclude. □

5. Compactness

In this section we want to prove Theorem 1.4.

Proof of Theorem 1.4. Let $\{u_n\}_n \subset W^{1,2}(\Omega;\mathbb{R}^M)$ be such that

$$\sup_{n\in\mathbb{N}} \mathcal{E}_n(u_n) = C < \infty.$$

In the course of the Liminf proof, we achieve the following uniform bound without using any information about the convergence:

$$\sup_n \int_\Omega 2\sqrt{W^\eta(u_n)}|\nabla u_n|\,dx \leq C$$

Since W^η still only has two wells a, b, we can apply the classical Young Measure techniques in [19] to extract a subsequence such that $u_n \to u \in BV(\Omega;\{a,b\})$ strongly in L^2 (it is upgraded from strong L^1 to strong L^2 due to our quadratic coercivity).

□

6. Limsup inequality

The goal of this section is to prove the following result.

Proposition 6.1. *Let $u \in BV(\Omega;\{a,b\})$. Then, there exists a sequence $\{u_n\}_n \subset W^{1,2}(\Omega;\mathbb{R}^M)$ with $u_n \to u$ strongly in $L^2(\Omega;\mathbb{R}^M)$ such that*

$$\limsup_{n\to\infty} \mathcal{E}_n(u_n) \leq \mathcal{E}_\infty(u).$$

The proof of the limsup inequality requires two technical results that are well known to the experts. For the reader's convenience, we state them in here.

The first is an approximation result, stating that C^2 sets are dense both in configuration and in energy. This result is contained in the proof of [7, Lemma 3.1].

Proposition 6.2. *Let $E \subset \Omega$ be a set with finite perimeter. Then, there exists a sequence of sets $\{E_n\}_n$, where each $E_n \subset \Omega$ satisfies*

- *$\partial E_n \cap \Omega$ is of class C^2;*
- *$\mathcal{H}^{N-1}(\partial\Omega \cap \partial E_n) = 0$,*

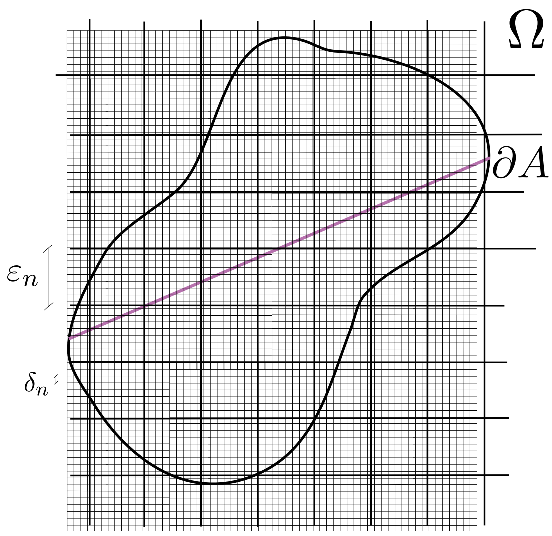

FIGURE 2. The two different scales ε_n and δ_n.

such that $E_n \to E$ with respect to the L^1 topology, and

$$\lim_{n\to\infty} \mathcal{E}_\infty(u_n) = \mathcal{E}_\infty(u),$$

where $u_n := \mathbb{1}_{E_n}$, and $u := \mathbb{1}_E$.

Next result ensures that, up to an error, it is possible to reparametrize a curve in such a way that the energy functional is bounded by the limiting energy. This was originally used in the article by Modica (see [27, Proof of Proposition 2]). Here we state the version used in [15, Lemma 4.5], since it states clearly the estimates that we will need. Note that none of the assumptions on W required in [15] are actually used, other than continuity in the second variable. Moreover, the lower bound for τ follows easily from the definition of τ given in the proof of the result.

Lemma 6.3. *Fix $\lambda > 0$, $\varepsilon > 0$. Let $\gamma \in C^1([-1,1];\mathbb{R}^M)$, with $\gamma(-1) = a$, $\gamma(1) = b$, and $\gamma'(s) \neq 0$ for all $s \in (-1,1)$. Then, there exist $\tau > 0$, and $C > 0$, with*

$$C\varepsilon \leq \tau \leq \frac{\varepsilon}{\sqrt{\lambda}} \int_{-1}^{1} |\gamma'(t)|\, dt,$$

and $g \in C^1((-\tau,\tau);[-1,1])$ such that

$$(g'(t))^2 = \frac{\lambda + W(x,\gamma(g(t)))}{\varepsilon^2|\gamma'(g(t))|^2}$$

for all $t \in (-\tau,\tau)$, $g(-\tau) = -1$, $g(\tau) = 1$, and

$$\int_{-\tau}^{\tau} \left[\frac{1}{\varepsilon} W_{\mathrm{hom}}\left(\gamma(g(t))\right) + \varepsilon|\gamma'\left(g(t)\right)|^2 \left(g'(t)\right)^2\right] dt$$
$$\leq \int_{-1}^{1} 2\sqrt{W_{\mathrm{hom}}(\gamma(s))}|\gamma'(s)|\, ds + 2\sqrt{\lambda}\int_{-1}^{1} |\gamma'(s)|\, ds.$$

We are now in position to prove the existence of a recovery sequence.

Proof of 6.1. Let $A := \{u = a\}$.

Step 1. By using Proposition 6.2 and a diagonalization argument, we get that it suffices to prove the result for $u \in BV(\Omega;\{a,b\})$ such that $\partial A \cap \Omega$ is of class C^2, and $\mathcal{H}^{N-1}(\partial A \cap \partial\Omega) = 0$.

Step 2. Let $u \in BV(\Omega;\{a,b\})$ be as in Step 1. Fix $\eta > 0$. Let $\gamma \in C^1([-1,1];\mathbb{R}^N)$ with $\gamma(-1) = a$, and $\gamma(1) = b$ be such that

$$\int_{-1}^{1} 2\sqrt{W_{\mathrm{hom}}(\gamma(t))}|\gamma'(t)|dt \leq \sigma_{\mathrm{hom}} + \eta. \tag{6.1}$$

Without loss of generality, we can assume that $\gamma'(s) \neq 0$ for all $s \in (-1,1)$. For each $n \in \mathbb{N}$, let $\tau_n > 0$, and $g_n \in C^1([0,\tau_n];\mathbb{R}^M)$ be those given by Lemma 6.3 relative to the choice of

$$\varepsilon = \varepsilon_n, \qquad \lambda = \left(\frac{\eta}{L(\gamma)}\right)^2, \tag{6.2}$$

where

$$L(\gamma) \coloneqq \int_{-1}^{1} |\gamma'(s)|\, ds < \infty.$$

Let $\mathrm{dist}(\cdot,\partial A) : \mathbb{R}^N \to \mathbb{R}$ be the signed distance function from ∂A. Note that, $\mathrm{dist}(\cdot,\partial A)$ is of class C^1, since ∂A is of class C^2. For $n \in \mathbb{N}$, define

$$u_n(x) \coloneqq \begin{cases} b & \mathrm{dist}(x,\partial A) > \tau_n, \\ \gamma\left(g_n(\mathrm{dist}(x,\partial A))\right) & |\mathrm{dist}(x,\partial A)| \leq \tau_n, \\ a & \mathrm{dist}(x,\partial A) < -\tau_n. \end{cases} \tag{6.3}$$

We claim that the sequence $\{u_n\}_n$ satisfies the required properties, up to an error η. First of all, we note that each $u_n \in W^{1,2}(\Omega;\mathbb{R}^M)$. Moreover, by using the fact that $\tau_n \to 0$ as $n \to \infty$, it is easy to see that $u_n \to u$ strongly in $L^2(\Omega;\mathbb{R}^M)$.

To prove the convergence of the energies, we argue as follows. Define

$$A_n \coloneqq \{x \in \Omega \,:\, |\mathrm{dist}(x,\partial A)| \leq \tau_n\}.$$

We note that

$$|A_n| \leq \mathcal{H}^{N-1}(\partial A)\tau_n. \tag{6.4}$$

For each $n \in \mathbb{N}$, consider the set of indexes

$$I_n \coloneqq \{i \in \mathbb{N} \,:\, z_i + \delta_n Q \subset A_n,\, z_i \in \delta_n \mathbb{Z}^N\},$$

$$B_n \coloneqq \{i \in \mathbb{N} \,:\, z_i + \delta_n Q \cap A_n \neq \emptyset,\, z_i \in \delta_n \mathbb{Z}^N\} \setminus I_n.$$

Let $K_n \coloneqq \# I_n$. Note that

$$\lim_{n\to\infty} K_n \left(\left\lfloor \frac{|A_n|}{\delta_n^N} \right\rfloor \right)^{-1} = 1. \tag{6.5}$$

Write

$$A_n \coloneqq \bigcup_{i \in I_n} (z_i + \delta_n Q) \cup R_n,$$

where

$$R_n \coloneqq \bigcup_{j \in B_n} (z_j + \delta_n Q) \cap A.$$

Then,

$$|R_n| \leq C_A \delta_n, \tag{6.6}$$

where $C_A > 0$ is a constant depending on ∂A.

Step 3. We claim that there exists a dimensional constant $C_N > 0$ such that

$$|u_n(x) - u_n(z_i)| \leq \omega_\gamma\left(C_N \frac{\delta_n}{\varepsilon_n}\right), \tag{6.7}$$

for all $x \in z_i + \delta_n Q$, and all $n \in \mathbb{N}$, and $i \in I_n$, where $\omega_\gamma : [0,\infty) \to [0,\infty)$ is the modulus of continuity of γ. Indeed,

$$\begin{aligned}
|u_n(x) - u_n(z_i)| &= |\gamma\left(g_n(\mathrm{dist}(x,\partial A))\right) - \gamma\left(g_n(\mathrm{dist}(z_i,\partial A))\right)| \\
&\leq \omega_\gamma\left(\frac{1}{\varepsilon_n}(|\mathrm{dist}(x,\partial A) - \mathrm{dist}(z_i,\partial A)|)\right) \\
&\leq \omega_\gamma\left(\frac{1}{\varepsilon_n}(|x - z_i|)\right) \\
&\leq \omega_\gamma\left(C_N \frac{\delta_n}{\varepsilon_n}\right),
\end{aligned}$$

where in the second step we used the fact that $|g_n'| \leq C/\varepsilon_n$.

Step 4. We claim that

$$\lim_{n\to\infty}\left|\frac{1}{\varepsilon_n}\int_{A_n}\left[W\left(\frac{x}{\delta_n}, u_n(x)\right) - W_{\mathrm{hom}}(u_n(x))\right]\,dx\right| = 0.$$

By using the unfolding operator restricted to A_n we get:

$$\begin{aligned}
\int_{A_n} W\left(\frac{x}{\delta_n}, u_n(x)\right)\,dx &= \sum_{i=1}^{K_n}\int_{z_i+\delta_n Q}\int_Q W(y, \mathcal{U}_{\delta_n} u_n)\,dydx + \int_{R_n} W\left(\frac{x}{\delta_n}, u_n(x)\right)\,dx \\
&= \delta_n^N \sum_{i=1}^{K_n}\int_Q W(y, u_n(z_i + \delta_n y))\,dy\; + \int_{R_n} W\left(\frac{x}{\delta_n}, u_n(x)\right)\,dx.
\end{aligned}$$

Thus, we can write

$$\begin{aligned}
&\int_{A_n}\left[W\left(\frac{x}{\delta_n}, u_n(x)\right) - W_{\mathrm{hom}}(u_n(x))\right]\,dx \\
&\quad = \delta_n^N \sum_{i=1}^{K_n}\left[\int_Q W(y, u_n(z_i+\delta_n y))\,dy - W_{\mathrm{hom}}(u_n(z_i))\right] \\
&\qquad\quad + \sum_{i=1}^{K_n}\left[\delta_n^N W_{\mathrm{hom}}(u_n(z_i)) - \int_{z_i+\delta_n Q} W_{\mathrm{hom}}(u_n(x))\,dx\right] \\
&\qquad\quad + \int_{R_n}\left[W\left(\frac{x}{\delta_n}, u_n(x)\right) - W_{\mathrm{hom}}(u_n(x))\right]\,dx \\
&\quad =: J_n^1 + J_n^2 + J_n^3.
\end{aligned} \tag{6.8}$$

We now estimate the three terms J_n^1, J_n^2, and J_n^3 separately. Since

$$\sup_{n\in\mathbb{N}} \|u_n\|_{L^\infty} \leq M < \infty, \tag{6.9}$$

we have that

$$|J_n^3| \leq C|R_n| \leq C\delta_n, \tag{6.10}$$

where in the last step we used (6.6).

Now we estimate J_n^2. Let $\omega_{\hom}$ be a modulus of continuity of $W_{\hom}$ in $B(0,M)$, where $M>0$ is the constant in (6.9). Then,

$$\begin{aligned}
|J_n^2| &= \left| \int_{\bigcup_{i=1}^{K_n} z_i+\delta_n Q} W_{\hom}(u_n(z_i)) - W_{\hom}(u_n(x))\,dx \right| \\
&\le \int_{\bigcup_{i=1}^{K_n} z_i+\delta_n Q} \omega_{\hom}(|u_n(z_i)-u_n(x)|)\,dx \\
&\le K_n\delta_n^N \omega_{\hom}\left(\omega_\gamma\left(C_N\frac{\delta_n}{\varepsilon_n}\right)\right) \\
&\le C\varepsilon_n\omega_{\hom}\left(\omega_\gamma\left(C_N\frac{\delta_n}{\varepsilon_n}\right)\right), \qquad (6.11)
\end{aligned}$$

where in the previous to last step we used Step 3, while in the last step we used (6.4) together with (6.5).

Finally, we estimate J_n^1. By using the definition of $W_{\hom}$, we write

$$\int_Q W(y,u_n(z_i+\delta_n y))\,dy - W_{\hom}(u_n(z_i)) = \int_Q \left[W(y,u_n(z_i+\delta_n y)) - W(y,u_n(z_i))\right]\,dy.$$

Thus, by using a similar argument to that in (6.11), we obtain

$$|J_n^1| \le C\varepsilon_n\omega_W\left(\omega_\gamma\left(C_N\frac{\delta_n}{\varepsilon_n}\right)\right), \qquad (6.12)$$

where ω_W is a modulus of continuity of W in $B(0,M)$, where $M>0$ is the constant in (6.9).

By combining (6.12)(6.11)(6.10), we get

$$\begin{aligned}
&\left| \frac{1}{\varepsilon_n}\int_{A_n} W\left(\frac{x}{\delta_n},u_n(x)\right) - W_{\hom}(u_n(x))\,dx \right| \\
&\qquad \le \frac{1}{\varepsilon_n}\left(|J_n^1|+|J_n^2|+|J_n^3|\right) \\
&\qquad \le C\left(\omega_W\left(\omega_\gamma\left(C\frac{\delta_n}{\varepsilon_n}\right)\right) + \omega_{\hom}\left(\omega_\gamma\left(C\frac{\delta_n}{\varepsilon_n}\right)\right) + \frac{\delta_n}{\varepsilon_n}\right) \\
&\qquad \to 0, \qquad (6.13)
\end{aligned}$$

as $n\to\infty$, where in the last step we used the fact that $\delta_n \ll \varepsilon_n$, together with $\lim_{t\to 0}\omega_W(t) = \lim_{t\to 0}\omega_{\hom}(t) = 0$.

Step 5. We conclude as follows. By using step 4, and the coarea formula (see [1, Theorem 2.93 and Remark 2.94]), we get that

$$
\begin{aligned}
\limsup_{n\to\infty} \mathcal{E}_n(u_n) &= \limsup_{n\to\infty} \int_\Omega \left[\frac{1}{\varepsilon_n} W\left(\frac{x}{\delta_n}, u_n\right) + \varepsilon_n |\nabla u_n|^2\right] dx \\
&\le \limsup_{n\to\infty} \int_\Omega \left[\frac{1}{\varepsilon_n} W_{\mathrm{hom}}(u_n) + \varepsilon_n |\nabla u_n|^2\right] dx \\
&\qquad + \limsup_{n\to\infty} \left| \frac{1}{\varepsilon_n} \int_\Omega \left[W\left(\frac{x}{\delta_n}, u_n(x)\right) - W_{\mathrm{hom}}(u_n(x))\right] dx \right| \\
&= \limsup_{n\to\infty} \int_\Omega \left[\frac{1}{\varepsilon_n} W_{\mathrm{hom}}(u_n) + \varepsilon_n |\nabla u_n|^2\right] dx \\
&= \limsup_{n\to\infty} \int_{-\tau_n}^{\tau_n} \left[\frac{1}{\varepsilon_n} W_{\mathrm{hom}}(\gamma(g_n(s))) + \varepsilon_n |\gamma'(g_n(s)) g_n'(s)|^2\right] \cdot \\
&\qquad \cdot \mathcal{H}^{N-1}(\{x\in\Omega : \mathrm{dist}(x,\partial A) = s\})\, ds \\
&\le \limsup_{n\to\infty} \sup_{|s|\le\tau_n} \mathcal{H}^{N-1}(\{x\in\Omega : \mathrm{dist}(x,\partial A) = s\}) \cdot \\
&\qquad \cdot \int_{-\tau_n}^{\tau_n} \left[\frac{1}{\varepsilon_n} W_{\mathrm{hom}}(\gamma(g_n(s))) + \varepsilon_n |\gamma'(g_n(s)) g_n'(s)|^2\right] ds \\
&\le \limsup_{n\to\infty} \sup_{|s|\le\tau_n} \mathcal{H}^{N-1}(\{x\in\Omega : \mathrm{dist}(x,\partial A) = s\}) \cdot \\
&\qquad \cdot \left[\int_{-1}^{1} 2\sqrt{W_{\mathrm{hom}}(\gamma(t))} |\gamma'(t)| dt + 2\sqrt{\lambda} L(\gamma) \right] \\
&\le \mathcal{H}^{N-1}(\partial A)[\sigma_{\mathrm{hom}} + 3\eta],
\end{aligned}
$$

where last step follows from (6.13), together with (6.2), Lemma 6.3, and the fact that, since ∂A is of class C^2, it holds

$$
\lim_{n\to\infty} \mathcal{H}^{N-1}(\{x\in\Omega : \mathrm{dist}(x,\partial A) = s\}) = \mathcal{H}^{N-1}(\partial A).
$$

Since $\eta > 0$ is arbitrary, we conclude. □

7. The mass constrained functional

As it is usually the case, the strategy of the proof for the Gamma-limit in the unconstrained case are stable enough to be used for the mass constrained functional, with minor changes. The liminf inequality follows from exactly the same proof. What needs to be checked is that, in the construction of the recovery sequence, it is possible to adjust the mass of the sets constructed in the proof of Proposition 6.1. There are some standard ways to do that, that can be found in the paper by Fonseca and Tartar (see [19]), in the paper by Baldo (see [7, Lemma 3.3], although the argument there is not correct), and in the paper by Ishige (see [23]). Unfortunately, such strategies use the assumption that the regularity of the potential W, that we do not require in here. Luckily, there is another way to do it, that does not require such assumption. We report here for the reader's convenience (since we did not find any published source to refer to).

Lemma 7.1. *Fix $m > 0$. Let $u \in BV(\Omega; \{a,b\})$ with $|\{u = a\}| = m$. Then, there exists a sequence $\{u_n\}_n \subset W^{1,2}(\Omega;\mathbb{R}^M)$ with $u_n \to u$ strongly in $L^2(\Omega;\mathbb{R}^M)$ such that*

$$
\limsup_{n\to\infty} \widetilde{\mathcal{E}}_n(u_n) \le \widetilde{\mathcal{E}}_\infty(u).
$$

Proof. We show how to modify Proposition 6.2 and the definition of the function u_n in step 2 of the proof of Proposition 6.1 in order to get the recovery sequence for the mass constrained functional.

Step 1. Let $E := \{u = a\}$. Let $\{E_n\}_n$ be the sequence provided by Proposition 6.2. We would like to modify them in such a way that they have the required mass. The strategy we use is a variant of an idea by Ryan Murray: to modify E in such a way to find a ball that is always contained in the sequence of approximating sets, and one that is always outside the sequence of approximating sets.

Let $x_0, x_1 \in \Omega$ be points of density one and zero for E, respectively. Namely,

$$\lim_{r\to 0} \frac{|E \cap B(x_1, r)|}{|B(x_1, r)|} = 1, \qquad \lim_{r\to 0} \frac{|E \cap B(x_0, r)|}{|B(x_0, r)|} = 0.$$

Then, there exists $R > 0$ such that

$$\frac{3}{4} \le \frac{|E \cap B(x_1, r)|}{|B(x_1, r)|} \le 1, \qquad 0 \le \frac{|E \cap B(x_0, r)|}{|B(x_0, r)|} \le \frac{1}{4}, \tag{7.1}$$

for all $r \le R$. Without loss of generality, up to decreasing the value of $R > 0$, we can assume that $B(x_1, r) \cap B(x_0, r) = \emptyset$. For $r \le R$, let

$$\widetilde{E}_r := E \cup B(x_1, r) \setminus B(x_0, r).$$

Note that

$$\lim_{r\to 0} |P(E_r; \Omega) - P(E; \Omega)| = 0, \qquad \|\mathbb{1}_{E_r} - \mathbb{1}_E\|_{L^1(\Omega)} = 0.$$

Let $(F^r_n)_n$ be the sequence of sets given by Proposition 6.2 relative to E_r. Then, for n large, we get that

$$|P(F^r_n; \Omega) - P(E_r; \Omega)| < r, \qquad \|\mathbb{1}_{E^r_n} - \mathbb{1}_{E^r}\|_{L^1(\Omega)} < r.$$

Since such sets F^r_n's are obtained with the standard procedure of mollification of the characteristic function of E_r, and then by taking a super level set, we get that there exists $\bar{n} \in \mathbb{N}$ such that

$$B\left(x_0, \left(\frac{4}{5}\right)^N r\right) \subset \Omega \setminus F^r_n, \qquad B\left(x_1, \left(\frac{4}{5}\right)^N r\right) \subset F^r_n,$$

for all $n \ge \bar{n}$. Now, assume that $|F^r_n| < m$. Let $s_n > 0$ be such that $|F^r_n| + |B(0, s_n)| = m$. We claim that

$$s_n < \left(\frac{4}{5}\right)^N r,$$

for all $n \ge \bar{n}$. Indeed, for n large enough, by using (7.1) it holds that

$$|E_r| - |E| < \frac{3}{4}|B(x_1, r)|.$$

Therefore, by considering the set

$$\widetilde{F}^r_n := F^r_n \cup B(x_0, s_n),$$

we get that $\widetilde{F}^r_n$ is of class C^2. A similar argument is used to fix the mass in the case where $|F^r_n| > m$. This sequence satisfies the required properties.

Step 2. It is easy to see that, for each $n \in \mathbb{N}$, there exists $v_n \in \mathbb{R}$ such that the function

$$\widetilde{u}_n(x) := \begin{cases} b & \mathrm{dist}(x, \partial A) > \tau_n, \\ \gamma\left(g_n(\mathrm{dist}(x, \partial A) + v_n)\right) & |\mathrm{dist}(x, \partial A)| \leq \tau_n, \\ a & \mathrm{dist}(x, \partial A) < -\tau_n. \end{cases}$$

is such that

$$\int_\Omega u_n(x)\, dx = ma + (1-m)b.$$

By using the fact that $v_n \to 0$ as $n \to \infty$, it is possible to check that all of the steps in the proof of Proposition 6.1 can be carried out in a similar way. This allows to conclude. $\square$

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